Proof by Induction

Exercise A

1 Prove by induction that for any positive integer n, $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$. **(5 marks)**

2 Prove by induction that for any positive integer n, $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$. **(5 marks)**

3 a Prove by induction that for any positive integer n:

$$\sum_{r=1}^{n} r(r-1) = \frac{1}{3}n(n+1)(n-1)$$ **(6 marks)**

b Hence deduce an expression, in terms of n, for $\sum_{r=1}^{2n+1} r(r-1)$. **(3 marks)**

4 a Prove by induction that, for any positive integer n:

$$\sum_{r=1}^{n} r(3r-1) = n^2(n+1)$$ **(6 marks)**

b Hence use the standard result for $\sum_{r=1}^{n} r^3$ to find a value of n such that $\sum_{r=1}^{n} r^3 = 4\sum_{r=1}^{n} r(3r-1)$. **(5 marks)**

5 Prove by induction that for any positive integer n,

a $\sum_{r=1}^{n} \left(\frac{1}{2}\right)^r = 1 - \frac{1}{2^n}$ **b** $\sum_{r=1}^{n} r(r!) = (n+1)! - 1$ **c** $\sum_{r=1}^{n} \frac{4}{r(r+2)} = \frac{n(3n+5)}{(n+1)(n+2)}$

6 The box below shows a student's attempts to prove $\left(\sum_{r=1}^{n} r\right)^2 = \sum_{r=1}^{n} r^2$ using induction.

Let $n = 1$. Then LHS $= \left(\sum_{r=1}^{1} r^2\right) = (1)^2 = 1$, and RHS $= \sum_{r=1}^{1} r^2 = 1^2 = 1$, so that LHS = RHS (Basis step).

Now we assume the statement is true for $n = k$:

$$\left(\sum_{r=1}^{k} r\right)^2 = \sum_{r=1}^{k} r^2$$

and so for $n = k + 1$ the statement is

$$\left(\sum_{r=1}^{k+1} r\right)^2 = \sum_{r=1}^{k+1} r^2$$

Hence, by the principle of mathematical induction, the statement is true for all $n \in \mathbb{Z}^+$.

a Identify the error made in the proof. **(2 marks)**

b Give a counter-example to show that the original statement is not true. **(1 mark)**

Question 7 is on the next page.

7 A student claims that $\sum_{r=1}^{n} r = \frac{1}{2}(n^2 + n + 1)$, and produces the following proof.

> Assume that the statement is true for $n = k$:
>
> $$\sum_{r=1}^{k} r = \frac{1}{2}(k^2 + k + 1)$$
>
> When $n = k + 1$:
>
> $$\begin{aligned}\sum_{r=1}^{k+1} r &= \sum_{r=1}^{k} r + (k + 1)\\ &= \frac{1}{2}(k^2 + k + 1) + (k + 1)\\ &= \frac{1}{2}(k^2 + k + 1 + 2(k + 1))\\ &= \frac{1}{2}((k^2 + 2k + 1) + (k + 1) + 1)\\ &= \frac{1}{2}((k + 1)^2 + (k + 1) + 1)\end{aligned}$$
>
> This is the original formula but with $n = k + 1$. Hence, by the principle of mathematical induction, the statement is true for all $n \in \mathbb{Z}^+$.

a Identify the error made in the proof. **(2 marks)**

b Give a counter-example to show that the original statement is not true. **(1 mark)**

Exercise B

1 Prove by induction that for all positive integers n:

a $8^n - 1$ is divisible by 7

b $3^{2n} - 1$ is divisible by 8

c $5^n + 9^n + 2$ is divisible by 4

d $2^{4n} - 1$ is divisible by 15

e $3^{2n-1} + 1$ is divisible by 4

f $n^3 + 6n^2 + 8n$ is divisible by 3

g $n^3 + 5n$ is divisible by 6

h $2^n(3^{2n}) - 1$ is divisible by 17

2 $f(n) = 13^n - 6^n$

a Show that $f(k + 1) = 6f(k) + 7(13^k)$. **(3 marks)**

b Hence, or otherwise, prove by induction that for all positive integers n, $f(n)$ is divisible by 7. **(4 marks)**

3 $g(n) = 5^{2n} - 6n + 8$

a Show that $g(k + 1) = 25g(k) + 9(16k - 22)$. **(3 marks)**

b Hence, or otherwise, prove by induction that for all positive integers n, $g(n)$ is divisible by 9. **(4 marks)**

4 Prove by induction that for all positive integers n, $8^n - 3^n$ is divisible by 5. **(6 marks)**

5 Prove by induction that for all positive integers n, $3^{2n+2} + 8n - 9$ is divisible by 8. **(6 marks)**

6 Prove by induction that for all positive integers n, $2^{6n} + 3^{2n-2}$ is divisible by 5. **(6 marks)**

Exercise C

1 Prove by induction that for all positive integers n,

$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}^n = \begin{pmatrix}1 & 2n\\0 & 1\end{pmatrix}$ **(6 marks)**

2 Prove by induction that for all positive integers n,

$\begin{pmatrix}3 & -4\\1 & -1\end{pmatrix}^n = \begin{pmatrix}2n+1 & -4n\\n & -2n+1\end{pmatrix}$ **(6 marks)**

3 Prove by induction that for all positive integers n,

$\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}^n = \begin{pmatrix}2^n & 0\\2^n-1 & 1\end{pmatrix}$ **(6 marks)**

4 a Prove by induction that for all positive integers n,

$\begin{pmatrix}5 & -8\\2 & -3\end{pmatrix}^n = \begin{pmatrix}4n+1 & -8n\\2n & 1-4n\end{pmatrix}$ **(6 marks)**

b Hence find the value of n such that:

$\begin{pmatrix}-1 & 3\\-2 & 5\end{pmatrix}\begin{pmatrix}5 & -8\\2 & -3\end{pmatrix}^n = \begin{pmatrix}11 & -21\\10 & -19\end{pmatrix}$ **(4 marks)**

5 The matrix $\mathbf{M} = \begin{pmatrix}2 & 5\\0 & 1\end{pmatrix}$

a Prove by induction that for all positive integers n,

$\mathbf{M}^n = \begin{pmatrix}2^n & 5(2^n-1)\\0 & 1\end{pmatrix}$ **(6 marks)**

b Hence find an expression for $(\mathbf{M}^n)^{-1}$ in terms of n. **(4 marks)**

Exercise D

1 Prove by induction that $9^n - 1$ is divisible by 8 for all positive integers n. **(6 marks)**

2 The matrix $\mathbf{B}$ is given by $\mathbf{B} = \begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}$.

a Find $\mathbf{B}^2$ and $\mathbf{B}^3$.

b Use your answer to part **a** to suggest a general statement for $\mathbf{B}^n$, for all positive integers n.

c Prove by induction that your answer to part **b** is correct.

3 Prove by induction that for all positive integers n, $\sum_{r=1}^{n}(3r+4) = \frac{1}{2}n(3n+11)$. **(6 marks)**

4 The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix}9 & 16\\-4 & -7\end{pmatrix}$.

a Prove by induction that $\mathbf{A}^n = \begin{pmatrix}8n+1 & 16n\\-4n & 1-8n\end{pmatrix}$ for all positive integers n. **(6 marks)**

The matrix $\mathbf{B}$ is given by $\mathbf{B} = (\mathbf{A}^n)^{-1}$.

b Hence find $\mathbf{B}$ in terms of n. **(4 marks)**

5 The function f is defined by $f(n) = 5^{2n-1} + 1$, where n is a positive integer.

a Show that $f(n+1) - f(n) = \mu(5^{2n-1})$, where μ is an integer to be determined. **(3 marks)**

b Hence prove by induction that $f(n)$ is divisible by 6. **(4 marks)**

6 Prove by induction that $7^n + 4^n + 1$ is divisible by 6 for all positive integers n. **(6 marks)**

7 Prove by induction that for all positive integers n, $\sum_{r=1}^{n} r(r+4) = \frac{1}{6}n(n+1)(2n+13)$. **(6 marks)**

8 **a** Prove by induction that for all positive integers n:

$$\sum_{r=1}^{2n} r^2 = \frac{1}{3}n(2n+1)(4n+1)$$ **(6 marks)**

b Given that $\sum_{r=1}^{2n} r^2 = k\sum_{r=1}^{n} r^2$, show that k must satisfy $n = \frac{2-k}{k-8}$ **(5 marks)**

9 The matrix $\mathbf{M} = \begin{pmatrix} 2c & 1 \\ 0 & c \end{pmatrix}$ for some positive constant c

a Prove by induction that for all positive integers n:

$$\mathbf{M}^n = c^n \begin{pmatrix} 2^n & \frac{2^n - 1}{c} \\ 0 & 1 \end{pmatrix}$$ **(7 marks)**

b Given that $\det(\mathbf{M}^n) = 50^n$, find the value of c. **(5 marks)**