Mathematics

Revision – Integration - Answers

1.

(i) **Way 1**	$\dfrac{3y-4}{y(3y+2)} \equiv \dfrac{A}{y} + \dfrac{B}{(3y+2)} \Rightarrow 3y-4 = A(3y+2) + By$	See notes	M1
	$y=0 \Rightarrow -4 = 2A \Rightarrow A = -2$	At least one of their $A=-2$ or their $B=9$	A1
	$y = -\frac{2}{3} \Rightarrow -6 = -\frac{2}{3}B \Rightarrow B = 9$	Both their $A=-2$ **and** their $B=9$	A1
	$\displaystyle\int \frac{3y-4}{y(3y+2)}\,\mathrm{d}y = \int \frac{-2}{y} + \frac{9}{(3y+2)}\,\mathrm{d}y$	Integrates to give at least one of **either** $\dfrac{A}{y} \to \pm\lambda\ln y$ **or** $\dfrac{B}{(3y+2)} \to \pm\mu\ln(3y+2)$, $A \neq 0$, $B \neq 0$	M1
		At least one term correctly followed through from their A or from their B	A1 ft
	$= -2\ln y + 3\ln(3y+2)\ \{+c\}$	$-2\ln y + 3\ln(3y+2)$ **or** $-2\ln y + 3\ln(y+\frac{2}{3})$ with correct bracketing, simplified or un-simplified. Can apply isw.	A1 cao
			[6]
(ii) (a) **Way 1**	$\left\{x = 4\sin^2\theta \Rightarrow\right\} \dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 8\sin\theta\cos\theta$ or $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 4\sin2\theta$ or $\mathrm{d}x = 8\sin\theta\cos\theta\,\mathrm{d}\theta$		B1
	$\displaystyle\int \sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}}\,.\,8\sin\theta\cos\theta\,\{\mathrm{d}\theta\}$ or $\displaystyle\int \sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}}\,.\,4\sin2\theta\,\{\mathrm{d}\theta\}$		M1
	$\displaystyle= \int \underline{\underline{\tan\theta}}\,.\,8\sin\theta\cos\theta\,\{\mathrm{d}\theta\}$ or $\displaystyle\int \underline{\underline{\tan\theta}}\,.\,4\sin2\theta\,\{\mathrm{d}\theta\}$	$\sqrt{\left(\dfrac{x}{4-x}\right)} \to \pm K\tan\theta$ or $\pm K\left(\dfrac{\sin\theta}{\cos\theta}\right)$	$\underline{\underline{\text{M1}}}$
	$\displaystyle= \int 8\sin^2\theta\,\mathrm{d}\theta$	$\displaystyle\int 8\sin^2\theta\,\mathrm{d}\theta$ including $\mathrm{d}\theta$	A1
	$3 = 4\sin^2\theta$ or $\dfrac{3}{4} = \sin^2\theta$ or $\sin\theta = \dfrac{\sqrt{3}}{2} \Rightarrow \theta = \dfrac{\pi}{3}$ $\{x = 0 \to \theta = 0\}$	Writes down a correct equation involving $x=3$ leading to $\theta = \dfrac{\pi}{3}$ and no incorrect work seen regarding limits	B1
			[5]
(ii) (b)	$\displaystyle= \{8\}\int\left(\frac{1-\cos2\theta}{2}\right)\mathrm{d}\theta \quad \left\{= \int(4-4\cos2\theta)\,\mathrm{d}\theta\right\}$	Applies $\cos2\theta = 1-2\sin^2\theta$ to their integral. (See notes)	M1
	$= \{8\}\left(\dfrac{1}{2}\theta - \dfrac{1}{4}\sin2\theta\right) \quad \{= 4\theta - 2\sin2\theta\}$	For $\pm\alpha\theta \pm \beta\sin2\theta$, $\alpha, \beta \neq 0$	M1
		$\sin^2\theta \to \left(\dfrac{1}{2}\theta - \dfrac{1}{4}\sin2\theta\right)$	A1
	$\displaystyle\left\{\int_0^{\frac{\pi}{3}} 8\sin^2\theta\,\mathrm{d}\theta = 8\left[\frac{1}{2}\theta - \frac{1}{4}\sin2\theta\right]_0^{\frac{\pi}{3}}\right\} = 8\left(\left(\frac{\pi}{6} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\right) - (0+0)\right)$		
	$= \dfrac{4}{3}\pi - \sqrt{3}$	"two term" exact answer of e.g. $\dfrac{4}{3}\pi - \sqrt{3}$ or $\dfrac{1}{3}(4\pi - 3\sqrt{3})$	A1 o.e.
			[4]

2.

(a)	$A = \int_0^3 \sqrt{(3-x)(x+1)}\ \mathrm{d}x\ ,\ \ x = 1 + 2\sin\theta$		
	$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\cos\theta$	$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\cos\theta$ or $2\cos\theta$ used correctly in their working. Can be implied.	B1
	$\left\{\int \sqrt{(3-x)(x+1)}\ \mathrm{d}x \text{ or } \int \sqrt{(3+2x-x^2)}\ \mathrm{d}x\right\}$		
	$= \int \sqrt{(3-(1+2\sin\theta))((1+2\sin\theta)+1)}\ \ 2\cos\theta\ \{\mathrm{d}\theta\}$	Substitutes for both x and $\mathrm{d}x$, where $\mathrm{d}x \neq \lambda\mathrm{d}\theta$. Ignore $\mathrm{d}\theta$	M1
	$= \int \sqrt{(2-2\sin\theta)(2+2\sin\theta)}\ \ 2\cos\theta\ \{\mathrm{d}\theta\}$ $= \int \sqrt{(4-4\sin^2\theta)}\ \ 2\cos\theta\ \{\mathrm{d}\theta\}$		
	$= \int \sqrt{(4-4(1-\cos^2\theta)}\ \ 2\cos\theta\ \{\mathrm{d}\theta\}$ or $\int \sqrt{4\cos^2\theta}\ \ 2\cos\theta\ \{\mathrm{d}\theta\}$	Applies $\cos^2\theta = 1 - \sin^2\theta$ **see notes**	M1
	$= 4\int \cos^2\theta\ \mathrm{d}\theta,\ \ \{k = 4\}$	$4\int \cos^2\theta\mathrm{d}\theta$ or $\int 4\cos^2\theta\mathrm{d}\theta$ **Note:** $\mathrm{d}\theta$ is required here.	A1
	$0 = 1 + 2\sin\theta$ or $-1 = 2\sin\theta$ or $\sin\theta = -\frac{1}{2} \Rightarrow \underline{\theta = -\frac{\pi}{6}}$ and $3 = 1 + 2\sin\theta$ or $2 = 2\sin\theta$ or $\sin\theta = 1 \Rightarrow \underline{\theta = \frac{\pi}{2}}$	See notes	B1
			[5]

(b)	$\left\{k\int \cos^2\theta\ \{\mathrm{d}\theta\}\right\} = \{k\}\int\left(\frac{1+\cos 2\theta}{2}\right)\{\mathrm{d}\theta\}$	Applies $\cos 2\theta = 2\cos^2\theta - 1$ to their integral	M1
	$= \{k\}\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)$	Integrates to give $\pm\alpha\theta \pm \beta\sin 2\theta,\ \alpha \neq 0,\ \beta \neq 0$ or $k(\pm\alpha\theta \pm \beta\sin 2\theta)$	M1 **(A1 on ePEN)**
	$\left\{\text{So } 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\mathrm{d}\theta = \left[2\theta + \sin 2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}}\right\}$ $= \left(2\left(\frac{\pi}{2}\right) + \sin\left(\frac{2\pi}{2}\right)\right) - \left(2\left(-\frac{\pi}{6}\right) + \sin\left(-\frac{2\pi}{6}\right)\right)$		
	$\left\{= (\pi) - \left(-\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right\} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$	$\frac{4\pi}{3} + \frac{\sqrt{3}}{2}$ or $\frac{1}{6}\left(8\pi + 3\sqrt{3}\right)$	A1 **cao cso**
			[3]

3.

$\left\{u=\sqrt{x} \Rightarrow\right\} \dfrac{du}{dx}=\dfrac{1}{2}x^{-\frac{1}{2}}$ or $\dfrac{dx}{du}=2u$	B1
$\displaystyle\int \frac{10}{2u^2+5u}.\,2u\ du$ Either $\left\{\int\right\}\dfrac{\pm ku}{\alpha u^2 \pm \beta u}\{du\}$ or $\left\{\int\right\}\dfrac{\pm k}{u(\alpha u^2 \pm \beta u)}\{du\}$	M1
$\left\{=\displaystyle\int \frac{20}{2u+5}\ du\right\}=\dfrac{20}{2}\ln(2u+5)$ — $\pm\lambda\ln(2u+5)$ or $\pm\lambda\ln\left(u+\dfrac{5}{2}\right)$, $\lambda \neq 0$ with no other terms.	M1
$\dfrac{20}{2u+5} \to \dfrac{20}{2}\ln(2u+5)$ or $10\ln\left(u+\dfrac{5}{2}\right)$	A1 cso
$\left\{\left[\dfrac{20}{2}\ln(2u+5)\right]_1^2\right\}=10\ln(2(2)+5)-10\ln(2(1)+5)$ — Substitutes limits of 2 and 1 in u (or 4 and 1 in x) and subtracts the correct way round.	M1
$10\ln 9-10\ln 7$ or $10\ln\left(\dfrac{9}{7}\right)$ or $20\ln 3-10\ln 7$	A1 oe cso
	[6]

4.

(i)	$\displaystyle\int xe^{4x}\ dx=\frac{1}{4}xe^{4x}-\int\frac{1}{4}e^{4x}\{dx\}$	$\pm\alpha xe^{4x}-\displaystyle\int\beta e^{4x}\{dx\}$, $\alpha\neq 0,\ \beta>0$	M1
		$\dfrac{1}{4}xe^{4x}-\displaystyle\int\frac{1}{4}e^{4x}\{dx\}$	A1
	$=\dfrac{1}{4}xe^{4x}-\dfrac{1}{16}e^{4x}\{+c\}$	$\dfrac{1}{4}xe^{4x}-\dfrac{1}{16}e^{4x}$	A1
			[3]
(ii)	$\displaystyle\int\frac{8}{(2x-1)^3}\ dx=\frac{8(2x-1)^{-2}}{(2)(-2)}\{+c\}$	$\pm\lambda(2x-1)^{-2}$	M1
		$\dfrac{8(2x-1)^{-2}}{(2)(-2)}$ or equivalent.	A1
	$\{=-2(2x-1)^{-2}\{+c\}\}$	*{Ignore subsequent working}.*	[2]
(iii)	$\dfrac{dy}{dx}=e^x\operatorname{cosec}2y\operatorname{cosec}y$ $\quad y=\dfrac{\pi}{6}$ at $x=0$		
	Main Scheme		
	$\displaystyle\int\frac{1}{\operatorname{cosec}2y\operatorname{cosec}y}\ dy=\int e^x\ dx$ or $\displaystyle\int\sin 2y\sin y\ dy=\int e^x\ dx$		B1 oe
	$\displaystyle\int 2\sin y\cos y\sin y\ dy=\int e^x\ dx$	Applying $\dfrac{1}{\operatorname{cosec}2y}$ or $\sin 2y \to 2\sin y\cos y$	M1
		Integrates to give $\pm\mu\sin^3 y$	M1
	$\dfrac{2}{3}\sin^3 y=e^x\{+c\}$	$2\sin^2 y\cos y \to \dfrac{2}{3}\sin^3 y$	A1
		$e^x \to e^x$	B1
	$\dfrac{2}{3}\sin^3\left(\dfrac{\pi}{6}\right)=e^0+c$ or $\dfrac{2}{3}\left(\dfrac{1}{8}\right)-1=c$	Use of $y=\dfrac{\pi}{6}$ and $x=0$ in an integrated equation containing c	M1
	$\left\{\Rightarrow c=-\dfrac{11}{12}\right\}$ giving $\dfrac{2}{3}\sin^3 y=e^x-\dfrac{11}{12}$	$\dfrac{2}{3}\sin^3 y=e^x-\dfrac{11}{12}$	A1
			[7]

	Alternative Method 1	
	$\int \frac{1}{\operatorname{cosec} 2y \operatorname{cosec} y}\, dy = \int e^x\, dx$ or $\int \sin 2y \sin y\, dy = \int e^x\, dx$	B1 oe
	$\int -\frac{1}{2}(\cos 3y - \cos y)\, dy = \int e^x\, dx$ — $\sin 2y \sin y \to \pm\lambda\cos 3y \pm \lambda\cos y$	M1
	Integrates to give $\pm\alpha\sin 3y \pm \beta\sin y$	M1
	$-\frac{1}{2}\left(\frac{1}{3}\sin 3y - \sin y\right) = e^x \{+c\}$ — $-\frac{1}{2}\left(\frac{1}{3}\sin 3y - \sin y\right)$	A1
	$e^x \to e^x$ *as part of solving their DE.*	B1
	$-\frac{1}{2}\left(\frac{1}{3}\sin\left(\frac{3\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)\right) = e^0 + c$ or $-\frac{1}{2}\left(\frac{1}{3} - \frac{1}{2}\right) - 1 = c$ — Use of $y = \frac{\pi}{6}$ and $x = 0$ in an integrated equation containing c	M1
	$\left\{\Rightarrow c = -\frac{11}{12}\right\}$ giving $-\frac{1}{6}\sin 3y + \frac{1}{2}\sin y = e^x - \frac{11}{12}$ — $-\frac{1}{6}\sin 3y + \frac{1}{2}\sin y = e^x - \frac{11}{12}$	A1
		[7]

5.

$\cos 8x$ seen in integrand	**M1**	
$F[x] = Ax + B\sin 8x$ oe	**M1***	A and B are non-zero constants
$F[x] = 6x - \frac{3}{8}\sin 8x$	**A1**	
$F[\frac{1}{8}\pi] - F[\frac{1}{16}\pi]$	**M1*dep**	
$\frac{3}{8}\pi + \frac{3}{8}$ oe	**A1**	
	[5]	

6.

$\frac{du}{dx} = 2x$ oe or $\frac{dx}{du} = \frac{1}{2}(u \pm 2)^{-1/2}$ oe	**M1**	
$\frac{Ax^2 + B}{2}$ or better from replacing dx NB $\frac{6x^3 + 4x}{2x} = \frac{6x^2 + 4}{2}$	**M1**	
substitution of $x^2 = u \pm 2$ or $x = (u \pm 2)^{1/2}$ in numerator	**M1**	**NB** $3(u+2) + 2$ or $3(u+2)^{3/2} + 2(u+2)^{1/2}$
$\int (\frac{3u + 8}{\sqrt{u}})[du]$ oe	**A1**	$\frac{3(u+2)+2}{\sqrt{u}}$ or better
$\frac{3u^{\frac{3}{2}}}{\frac{3}{2}} + \frac{8u^{\frac{1}{2}}}{\frac{1}{2}}$ oe	**A1**	or $6u^{3/2} + 16u^{1/2} - 4u^{3/2}$ from integration by parts
$2(x^2 - 2)^{\frac{3}{2}} + 16(x^2 - 2)^{\frac{1}{2}} + c$ cao	**A1** **[6]**	allow $2(x^2 - 2)^{1/2}(x^2 + 6) + c$ for final mark, **A0** if du not seen at some stage in the integral

7.

(i)	$\frac{A}{(x+4)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$	**B1**	
	$[16 + 5x - 2x^2] = A(x+1)^2 + B(x+1)(x+4) + C(x+4)$	**M1**	
	$A = -4$	**A1**	**NB** $36 = -9A$
	$C = 3$	**A1**	$9 = 3C$
	$B = 2$ isw	**A1**	$-2 = A + B,\ 5 = 2A + 5B + C$ $16 = A + 4B + 4C$
		[5]	NB $\frac{-4}{(x+4)} + \frac{2}{(x+1)} + \frac{3}{(x+1)^2}$

(ii)	$\int \frac{dy}{y} = \int \frac{16+5x-2x^2}{(x+1)^2(x+4)} dx$	B1	separation of variables
	$\frac{3}{(x+1)^2} + \frac{2}{(x+1)} - \frac{4}{(x+4)}$ seen in RHS, may be embedded	M1*	**FT** their partial fractions if two or three terms; ignore LHS
	$\frac{-3}{x+1} + 2\ln(x+1) - 4\ln(x+4) + c$	A1FT	**FT** their non-zero 3, 2 and 4; allow recovery from $x + 1^2$ in denominator; if brackets in log terms omitted, allow **A1** if recovery seen in substitution
	$\ln(\frac{1}{256}) = -3 + 2\ln 1 - 4\ln 4 + c$	M1*dep	substitution of $x = 0$ and $y = \frac{1}{256}$; allow if error in manipulation following integration;
	$c = 3$ cao	A1	or $A = e^{-3}$ from $y = Ae^{\frac{-3}{x+1}} \frac{(x+1)^2}{(x+4)^4}$
	$\ln y = \frac{-3}{2+1} + 2\ln(2+1) - 4\ln(2+4) + 3$	M1*dep	substitution of $x = 2$; dependent on award of previous **M1M1** and numerical value found for c
	$y = \frac{e^2}{144}$ oe	A1 [7]	

8.

(i)	$\frac{\sin x \times -\sin x - \cos x \times \cos x}{\sin^2 x}$ may be implied by $\frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$	M1	or $-\sin x \times \frac{1}{\sin x} + \cos x \times -(\sin x)^{-2} \times \cos x$ oe
	eg $= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$ and completion to $\frac{-1}{\sin^2 x}$ **AG**	A1	eg $= \frac{-\sin^2 x}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x}$ oe and completion to $\frac{-1}{\sin^2 x}$
		[2]	

(ii)		$\cos 2x = 2\cos^2 x - 1$ substituted in numerator	**M1**	or alternative form of double angle formula plus Pythagoras leading to no term in $\sin^2 x$ in numerator
		$\sin 2x = 2\sin x\cos x$ substituted in denominator	**M1**	
		$\dfrac{\sqrt{2}\cos x}{2\sin^2 x\cos x}$	**A1**	
		$F[x] = \pm k\dfrac{\cos x}{\sin x}$	**M1***	k must not be obtained from square rooting a negative number
		$F\left[\frac{1}{4}\pi\right] - F\left[\frac{1}{6}\pi\right]$	**M1dep***	eg $\dfrac{-\cos \pi/4}{\sqrt{2}\times\sin \pi/4} - \dfrac{-\cos \pi/6}{\sqrt{2}\times\sin \pi/6}$
		$= \frac{1}{2}(\sqrt{6} - \sqrt{2})$ www **AG**	**A1** [6]	

9.

$\dfrac{dt}{dx} = k(x+1)^{-\frac{1}{2}}$ or $\dfrac{dx}{dt} = 2t$ from $x = t^2 \pm 1$ oe	**M1**	or eg $k\,dt = \dfrac{dx}{\sqrt{x+1}}$ oe
$\int kte^{2t}\,dt$	**M1***	k is any non-zero constant
$kt\times\frac{1}{2}e^{2t} \pm k\int\frac{1}{2}e^{2t}\,dt$	**M1dep***	
$te^{2t} - \int e^{2t}\,dt$	**A1**	may be implied by the next **A1**
$te^{2t} - \frac{1}{2}e^{2t}$	**A1**	
$\sqrt{x+1}e^{2\sqrt{x+1}} - \frac{1}{2}e^{2\sqrt{x+1}} + c$ cao www	**A1** [6]	$+ c$ may be seen in previous line only for **A1**

10.

(i)		t^2 in quotient and $t^3 + 2t^2$ seen	B1	or $\frac{t(t^2-4)+4t}{(t+2)}$
		$-2t$ in quotient and $-2t^2 - (-2t^2 - 4t) = 4t$ seen	B1	$\frac{t(t+2)(t-2)}{(t+2)} + \frac{4t}{t+2}$
		completion to obtain correct quotient and remainder identified www	B1	$t(t-2) + \frac{4(t+2)-8}{t+2}$
			[3]	
(i)		alternatively $\frac{t^3}{t+2} \equiv At^2 + Bt + C + \frac{D}{(t+2)}$	B1	or $t^3 \equiv (At^2 + Bt + C)(t+2) + D$
		equate coefficients to obtain correctly $A = 1,\ 0 = 2A + B$ and $B = -2$ www	B1	
		$0 = 2B + C$ and $0 = 2C + D$ obtained and solved correctly www	B1	
			[3]	

(ii)		integration by parts with $u = \ln(t+2)$ and $dv = 6t^2$ to obtain $f(t) \pm \int g(t)(dt)$	M1*	$f(t)$ must include t^3 and $g(t)$ must **not** include a logarithm
		$2t^3\ln(t+2) - \int \frac{2t^3}{t+2}(dt)$ cao	A1	
		result from part (i) seen in integrand; must follow award of at least first M1	M1*	no integration required for this mark
		$F[t] = 2t^3\ln(t+2) \pm \frac{2t^3}{3} \pm 2t^2 \pm 8t \pm 16\ln(t+2)$	A1	$2t^3\ln(t+2) - \frac{2t^3}{3} + 2t^2 - 8t + 16\ln(t+2)$
		their $F[2] - F[1]$	M1dep*	at least one of their terms correctly integrated
		$-6\frac{2}{3} - 18\ln 3 + 32\ln 4$ oe cao	A1	
			[6]	

11.

$\frac{A}{1+2x}+\frac{B}{1-x}+\frac{C}{(1-x)^2}$ may be seen in later work	B1	or $\frac{A}{1+2x}+\frac{Bx+C}{(1-x)^2}$ may be seen later in later work
$2+x^2 \equiv A(1-x)^2+B(1+2x)(1-x)+C(1+2x)$	M1	or $A(1-x)^2+(Bx+C)(1+2x)$
$A=1$, $B=0$ and $C=1$ www	A1A1A1	
$\int\left(\frac{1}{1+2x}+\frac{1}{(1-x)^2}\right)\mathrm{d}x=$ $a\ln(1+2x)+b(1-x)^{-1}$	M1*	a and b are non-zero constants
$\mathrm{F}(x)=½\ln(1+2x)+(1-x)^{-1}$	A1	
their $\frac{1}{2}\ln(\frac{3}{2})+\frac{4}{3}-(\frac{1}{2}\ln 1+1)$	M1dep*	
$\frac{1}{2}\ln(\frac{3}{2})+\frac{4}{3}-0-1$	A1 [9]	and completion to given result www

12.

$u=\ln 3x$ and $\mathrm{d}v$ or $\frac{\mathrm{d}v}{\mathrm{d}x}=x^8$	M1	integ by parts as far as $\mathrm{f}(x)+/-\int \mathrm{g}(x)(\mathrm{d}x)$
$\frac{\mathrm{d}}{\mathrm{d}x}(\ln 3x)=\frac{1}{x}$ or $\frac{3}{3x}$	B1	stated or clearly used
$\frac{x^9}{9}\ln 3x-\int\frac{x^9}{9}$ their $\frac{\mathrm{d}u}{\mathrm{d}x}(\mathrm{d}x)$ FT	√A1	i.e. correct understanding of 'by parts'…
Indication that $\int kx^8\mathrm{d}x$ is required	M1	i.e. before integrating, product of terms must be taken
$\frac{x^9}{9}\ln 3x-\frac{x^9}{81}$ or $\frac{1}{9}x^9\left(\ln 3x-\frac{1}{9}\right)$ ISW (+c) <u>cao</u>	A1 [5]	$\frac{1}{9}\frac{x^9}{9}$ to be simplif to $\frac{x^9}{81}$; $\frac{3x^9}{243}$ satis
<u>If candidate manipulates ln(3x) first of all</u>		
$\ln(3x)=\ln 3+\ln x$	B1	
$u=\ln x$ and $\mathrm{d}v=x^8$	M1	In order to find $\int x^8\ln x\,\mathrm{d}x$:
$\frac{x^9}{9}\ln x-\int\frac{x^9}{9}.\frac{1}{x}(\mathrm{d}x)$ or better	A1	
$\frac{x^9}{9}\ln x-\frac{x^9}{81}$	A1	
Their $\int x^8\ln x\,\mathrm{d}x+\frac{x^9}{9}\ln 3$ (+ c) FT ISW	√A1	

13.

(i)		$\dfrac{(1+\tan x)-(1-\tan x)}{(1-\tan x)(1+\tan x)}$ $= \dfrac{2\tan x}{1-\tan^2 x} = \tan 2x$ Answer Given	M1 A1 [2]	Combine (or write as 2 separate fractions) using common denominator $\dfrac{2\tan x}{1-\tan^2 x}$ essential stage N.B. If $\tan x$ changed into $\frac{\sin x}{\cos x}$ before manipulation, apply same principles

(ii)		$\int \tan 2x \, dx = \lambda \ln(\sec 2x)$ or $\mu \ln(\cos 2x)$ [= F(x)]	M1	
		$\lambda = \dfrac{1}{2}$ or $\mu = -\dfrac{1}{2}$	A1	
		their $F[\dfrac{\pi}{6}]$ – their $F[\dfrac{\pi}{12}]$	M1	dependent on attempt at integration……..
		$\frac{1}{2}\ln 2 - \frac{1}{2}\ln\dfrac{2}{\sqrt{3}}$ oe	A1	i.e. any correct but probably unsimplified numerical version
		$\frac{1}{2}\ln\sqrt{3}$ or $\frac{1}{4}\ln 3$ or $\ln 3^{1/4}$ or $\frac{1}{2}\ln\dfrac{6}{2\sqrt{3}}$ oe ISW	+A1 [5]	i.e. any correct version in the form $a \ln b$

14.

Find du in terms of dx (or vv) or $\dfrac{du}{dx}$ or $\dfrac{dx}{du}$	M1	An attempt - not necessarily accurate
Substitute, changing given integral to $\int \dfrac{u-1}{u^2}(du)$	A1	No evidence of x at this A1 stage
Provided of form $\dfrac{au+b}{u^2}$, <u>either</u> split as $\dfrac{au}{u^2}+\dfrac{b}{u^2}$ …	M1	<u>or</u> use 'parts' with 'u' = $au+b$, 'dv' = $\dfrac{1}{u^2}$
Integrate as $\ln u + \dfrac{1}{u}$ or FT as $a \ln u - \dfrac{b}{u}$ [=F(u)]	√A1	or $-(au+b)\dfrac{1}{u} + a\ln u$ FT [=G(u)]
Re-substitute $u = 1 + \ln x$ in F(u)	M1	Re-substitute $u = 1 + \ln x$ in G(u)
$\ln(1+\ln x)+\dfrac{1}{1+\ln x}$ (+ c) ISW	A1 [6]	or $\ln(1+\ln x)-\dfrac{\ln x}{1+\ln x}$ (+ c) ISW

15.

$u = x$ and $dv = \cos 3x$	M1	integration by parts as far as $f(x) \pm \int g(x)\,dx$
$x \times \frac{1}{3}\sin 3x - \int \frac{1}{3}\sin 3x dx$	A2	**A1** for $x \times k \sin 3x - \int k \sin 3x\, dx$; $k \neq \frac{1}{3}$ or 0
$\frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x\, [+c]$ cao www ISW	A1 [4]	Not ….. $\frac{1}{3}\left(\frac{1}{3}\cos 3x\right)$ or ….. $- -\frac{1}{9}\cos 3x$

16.

Attempt diff to connect du & dx Correct result e.g. $\frac{du}{dx} = 2$ or $du = 2\,dx$	M1 A1	or find $\frac{du}{dx}$ or $\frac{dx}{du}$
Indef integ in terms of $u = \frac{1}{2}\int \frac{2u-3}{u^5}(du)$	A1	Must be completely in terms of u.
Integrate to $\frac{u^{-3}}{-3} - \frac{3u^{-4}}{-8}$ <u>oe</u>	A1A1	or (using 'by parts') $\frac{(2u-3)u^{-4}}{-8} - \frac{u^{-3}}{12}$
Use correct variable & correct values for limits	M1	Provided minimal attempt at $\int f(u)du$ made
$= \frac{-23}{384}$ oe $(-0.059895 \ldots)$	A1	Accept decimal answer only if minimum of first 3 marks scored
[ISW,e.g. changing to $\frac{23}{384}$]	[7]	

17.

(i)	I	$\frac{\cos x}{1+\sin x} - \frac{-\sin x}{\cos x}$ or $\frac{\cos x}{1+\sin x} + \frac{\sin x}{\cos x}$	B2	Each half (including 'middle' sign) scores B1
		$\frac{+/-\cos^2 x + /-\sin x(1+\sin x)}{(1+\sin x)\cos x}$	M1	Combine, provided derivative was of form $f'(x)/f(x)$
		$\frac{1+\sin x}{\cos x(1+\sin x)} = \frac{1}{\cos x}$ www **AG**	A1	$\cos^2 x + \sin^2 x = 1$ in intermediate step required
	II	Change to $\ln\left(\frac{1+\sin x}{\cos x}\right)$	B1	
		Change to $\ln(\sec x + \tan x)$	B1	Not $\ln(\frac{1}{\cos x} + \tan x)$
		Diff as $\frac{\text{attempt at } \frac{d}{dx}(\sec x + \tan x)}{\sec x + \tan x}$	M1	
		Reduce to $\sec x = \frac{1}{\cos x}$	A1	
	III	Change to $\ln\left(\frac{1+\sin x}{\cos x}\right)$	B1	
		Diff as $\frac{\text{attempt at quotient differentiation}}{\frac{1+\sin x}{\cos x}}$	M1	
		Fully correct differentiation	A1	
		Correct reduction to $\frac{1}{\cos x}$	A1 [4]	
(ii)		Indef integral = $\ln(1+\sin x) - \ln(\cos x)$ [Method I]	B1	or $\ln(\sec x + \tan x)$ [Method II]
		Substitute limits & use log manipulation	M1	Use of $\ln A - \ln B = \ln\frac{A}{B}$ anywhere in question
		Answer = $\ln(2+\sqrt{3})$	B1 [3]	Accept ln 3.73 or $\ln\frac{2+\sqrt{3}}{1}$ .....but not $\ln\frac{1+\sqrt{3}/2}{1/2}$

18.

(i)		Clear start to algebraic division (Quotient) = $x-1$ (Remainder) $= x+7$ Final answer: $x-1+\dfrac{x+7}{x^2-x-6}$	M1 A1 A1 A1 **[4]**	at least as far as x term in quot & subseq mult back final answer in correct form This must be shown in part (i) or, if not, then implied in part (ii) If no long division shown but only comparison of coefficients or otherwise, SR M0 B1 B1 B1
(ii)		Convert their $\dfrac{Cx+D}{x^2-x-6}$ to Partial Fracts $\dfrac{x+7}{x^2-x-6}=\dfrac{2}{x-3}-\dfrac{1}{x+2}$ Their …. $\int Ax+B\,\mathrm{d}x=\frac{1}{2}Ax^2+Bx$ or $\dfrac{(Ax+B)^2}{2A}$ $\int \dfrac{E}{x-3}+\dfrac{F}{x+2}\,\mathrm{d}x=E\ln(x-3)+F\ln(x+2)$ Using limits in a correct manner $8+\ln\dfrac{27}{4}$ $\left(8+\ln\dfrac{54}{8}\right)$ isw	M1 A1A1 B1 ft B1 ft M1 A1 **[7]**	 Correct fraction converted to correct PFs Tolerate some wrong signs provided intention clear Answer required in the form $a+\ln b$, so giving only a decimalised form is awarded A0