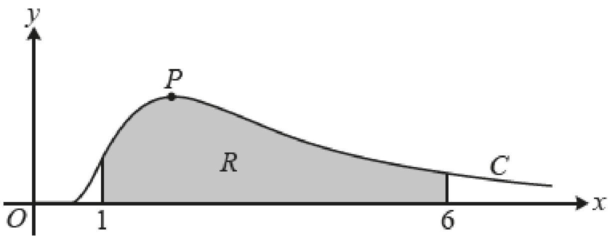
Mixed Exam Questions – Set 9

1.



The diagram shows the curve C with parametric equations

$x = \frac{3}{t}$, $y = t^3 e^{-2t}$, where $t > 0$.

The maximum point on C is denoted by P.

(a) Determine the exact coordinates of P. **[4]**

The shaded region R is enclosed by the curve, the x-axis and the lines $x = 1$ and $x = 6$.

(b) Show that the area of R is given by

$$\int_a^b 3te^{-2t}\,dt,$$

where a and b are constants to be determined. **[3]**

(c) Hence determine the exact area of R. **[5]**

2.

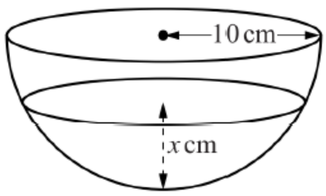


Fig. 9

Fig. 9 shows a hemispherical bowl, of radius 10 cm, filled with water to a depth of x cm. It can be shown that the volume of water, V cm^3, is given by

$$V = \pi(10x^2 - \tfrac{1}{3}x^3).$$

Water is poured into a leaking hemispherical bowl of radius 10 cm. Initially, the bowl is empty. After t seconds, the volume of water is changing at a rate, in cm^3 s^{-1}, given by the equation

$$\frac{dV}{dt} = k(20 - x),$$

where k is a constant.

(i) Find $\frac{dV}{dx}$, and hence show that $\pi x \frac{dx}{dt} = k$. **[4]**

(ii) Solve this differential equation, and hence show that the bowl fills completely after T seconds, where $T = \frac{50\pi}{k}$. **[5]**

Once the bowl is full, the supply of water to the bowl is switched off, and water then leaks out at a rate of $kx \text{ cm}^3 \text{s}^{-1}$.

(iii) Show that, t seconds later, $\pi(20 - x)\frac{dx}{dt} = -k$. **[3]**

(iv) Solve this differential equation.

Hence show that the bowl empties in $3T$ seconds. **[6]**

3.

[*In this question* **i** *and* **j** *are unit vectors due east and due north respectively. Position vectors are given relative to a fixed origin O.*]

Two ships P and Q are moving with constant velocities. Ship P moves with velocity $(2\mathbf{i} - 3\mathbf{j})$ km h^{-1} and ship Q moves with velocity $(3\mathbf{i} + 4\mathbf{j})$ km h^{-1}.

(a) Find, to the nearest degree, the bearing on which Q is moving.

(2)

At 2 pm, ship P is at the point with position vector $(\mathbf{i} + \mathbf{j})$ km and ship Q is at the point with position vector $(-2\mathbf{j})$ km.

At time t hours after 2 pm, the position vector of P is $\mathbf{p}$ km and the position vector of Q is $\mathbf{q}$ km.

(b) Write down expressions, in terms of t, for

(i) $\mathbf{p}$,

(ii) $\mathbf{q}$,

(iii) $\overrightarrow{PQ}$.

(5)

(c) Find the time when

(i) Q is due north of P,

(ii) Q is north-west of P.

(4)

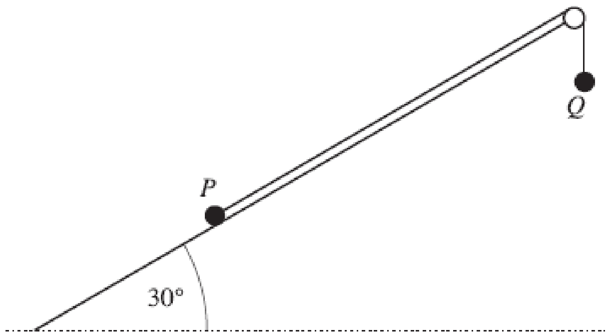
4.

The velocity $v \, \mathrm{m\,s^{-1}}$ of a car at time t s, during the first 20 s of its journey, is given by $v = kt + 0.03t^2$, where k is a constant. When $t = 20$ the acceleration of the car is $1.3 \, \mathrm{m\,s^{-2}}$. For $t > 20$ the car continues its journey with constant acceleration $1.3 \, \mathrm{m\,s^{-2}}$ until its speed reaches $25 \, \mathrm{m\,s^{-1}}$.

(i) Find the value of k. **[3]**

(ii) Find the total distance the car has travelled when its speed reaches $25 \, \mathrm{m\,s^{-1}}$. **[7]**

5.



Two particles P and Q are attached to opposite ends of a light inextensible string which passes over a small smooth pulley at the top of a rough plane inclined at 30° to the horizontal. P has mass 0.2 kg and is held at rest on the plane. Q has mass 0.2kg and hangs freely. The string is taut (see diagram). The coefficient of friction between P and the plane is 0.4. The particle P is released.

(i) State the tension in the string before P is released, and find the tension in the string after P is released. **[6]**

Q strikes the floor and remains at rest. P continues to move up the plane for a further distance of 0.8 m before it comes to rest. P does not reach the pulley.

(ii) Find the speed of the particles immediately before Q strikes the floor. **[5]**

(iii) Calculate the magnitude of the contact force exerted on P by the plane while P is in motion. **[3]**
