Mixed Exam Questions – Set 8

1.

(a) The function $\mathrm{f}(x)$ is defined for all values of x as $\mathrm{f}(x) = |ax - b|$, where a and b are positive constants.

(i) The graph of $y = \mathrm{f}(x) + c$, where c is a constant, has a vertex at (3, 1) and crosses the y-axis at (0, 7).

Find the values of a, b and c. **[3]**

(ii) Explain why $\mathrm{f}^{-1}(x)$ does not exist. **[1]**

(b) The function $\mathrm{g}(x)$ is defined for $x \geqslant \frac{q}{p}$ as $\mathrm{g}(x) = |px - q|$, where p and q are positive constants.

(i) Find, in terms of p and q, an expression for $\mathrm{g}^{-1}(x)$, stating the domain of $\mathrm{g}^{-1}(x)$. **[3]**

(ii) State the set of values of p for which the equation $\mathrm{g}(x) = \mathrm{g}^{-1}(x)$ has no solutions. **[1]**

2.

In this question you must show detailed reasoning.

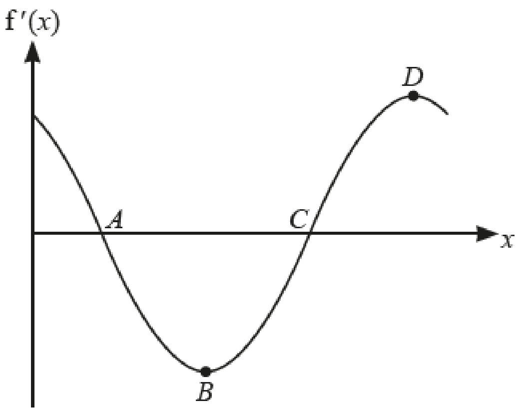
The function f is defined by $\mathrm{f}(x) = \cos x + \sqrt{3}\sin x$ with domain $0 \leqslant x \leqslant 2\pi$.

(a) Solve the following equations.

(i) $\mathrm{f}'(x) = 0$ **[4]**

(ii) $\mathrm{f}''(x) = 0$ **[3]**

The diagram shows the graph of the gradient function $y = \mathrm{f}'(x)$ for the domain $0 \leqslant x \leqslant 2\pi$.



(b) Use your answers to parts **(a)(i)** and **(a)(ii)** to find the coordinates of points A, B, C and D. **[2]**

(c) **(i)** Explain how to use the graph of the gradient function to find the values of x for which $f(x)$ is increasing. **[1]**

(ii) Using set notation, write down the set of values of x for which $f(x)$ is increasing in the domain $0 \leqslant x \leqslant 2\pi$. **[2]**

3.

A student wishes to prove that, for all positive integers a and b, $a^2 - 4b \neq 2$.

(a) Prove that $a^2 - 4b = 2 \Rightarrow a$ is even. **[2]**

(b) Hence or otherwise prove that, for all positive integers a and b, $a^2 - 4b \neq 2$. **[3]**

4.

A school contains 500 students in years 7 to 11 and 250 students in years 12 and 13. A random sample of 20 students is selected to represent the school at a parents' evening. The number of students in the sample who are from years 12 and 13 is denoted by X.

(a) State a suitable binomial model for X. **[1]**

Use your model to answer the following.

(b) **(i)** Write down an expression for $P(X = x)$. **[1]**

(ii) State, in set notation, the values of x for which your expression is valid. **[1]**

(c) Find $P(5 \leqslant X \leqslant 9)$. **[2]**

(d) State one disadvantage of using a random sample in this context. **[1]**

5.

A disease that affects trees shows no visible evidence for the first few years after the tree is infected.

A test has been developed to determine whether a particular tree has the disease. A positive result to the test suggests that the tree has the disease. However, the test is not 100% reliable, and a researcher uses the following model.

- If the tree has the disease, the probability of a positive result is 0.95.
- If the tree does not have the disease, the probability of a positive result is 0.1.

(a) It is known that in a certain county, A, 35% of the trees have the disease. A tree in county A is chosen at random and is tested.

Given that the result is positive, determine the probability that this tree has the disease. **[3]**

A forestry company wants to determine what proportion of trees in another county, B, have the disease. They choose a large random sample of trees in county B.

Each tree in the sample is tested and it is found that the result is positive for 43% of these trees.

(b) By carrying out a calculation, determine an estimate of the proportion of trees in county B that have the disease. **[4]**

6.

(a) "If m and n are irrational numbers, where $m \neq n$, then mn is also irrational."

Disprove this statement by means of a counter example.

(2)

(b) (i) Sketch the graph of $y = |x| + 3$

(ii) Explain why $|x| + 3 \geqslant |x + 3|$ for all real values of x.

(3)

7.

Given that θ is measured in radians, prove, from first principles, that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta) = -\sin\theta$$

You may assume the formula for $\cos(A \pm B)$ and that as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$

(5)

8.

A spherical mint of radius 5 mm is placed in the mouth and sucked.
Four minutes later, the radius of the mint is 3 mm.

In a simple model, the rate of decrease of the radius of the mint is inversely proportional to the square of the radius.

Using this model and all the information given,

(a) find an equation linking the radius of the mint and the time.
(You should define the variables that you use.)

(5)

(b) Hence find the total time taken for the mint to completely dissolve. Give your answer in minutes and seconds to the nearest second.

(2)

(c) Suggest a limitation of the model.

(1)

9.

A scientist is studying a population of mice on an island.

The number of mice, N, in the population, t months after the start of the study, is modelled by the equation

$$N = \frac{900}{3 + 7e^{-0.25t}}, \quad t \in \mathbb{R}, \quad t \geqslant 0$$

(a) Find the number of mice in the population at the start of the study.

(1)

(b) Show that the rate of growth $\frac{dN}{dt}$ is given by $\frac{dN}{dt} = \frac{N(300 - N)}{1200}$

(4)

The rate of growth is a maximum after T months.

(c) Find, according to the model, the value of T.

(4)

According to the model, the maximum number of mice on the island is P.

(d) State the value of P.

(1)

10.

It is given that $ABCD$ is a quadrilateral. The position vector of A is $\mathbf{i}+\mathbf{j}$, and the position vector of B is $3\mathbf{i}+5\mathbf{j}$.

(a) Find the length AB. **[1]**

(b) The position vector of C is $p\mathbf{i}+p\mathbf{j}$ where p is a constant greater than 1.

Given that the length AB is equal to the length BC, determine the position vector of C. **[3]**

(c) The point M is the midpoint of AC.

Given that $\overrightarrow{MD} = 2\overrightarrow{BM}$, determine the position vector of D. **[2]**

(d) State the name of the quadrilateral $ABCD$, giving a reason for your answer. **[2]**