Linear Transformations

Exercise A

1 Which of the following transformations are linear transformations?

a $P: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x \\ y+1 \end{pmatrix}$ b $Q: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ y \end{pmatrix}$ c $R: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x+y \\ x+xy \end{pmatrix}$

d $S: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -x \end{pmatrix}$ e $T: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y+3 \\ x+3 \end{pmatrix}$ f $U: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x \\ 3y-2x \end{pmatrix}$

2 Identify which of these are linear transformations and give their matrix representations. Give reasons to explain why the other transformations are not linear.

a $S: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x-y \\ 3x \end{pmatrix}$ b $T: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2y+1 \\ x-1 \end{pmatrix}$ c $U: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} xy \\ 0 \end{pmatrix}$

d $V: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2y \\ -x \end{pmatrix}$ e $W: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}$

3 Identify which of these are linear transformations and give their matrix representations. Give reasons to explain why the other transformations are not linear.

a $S: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$ b $T: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y \\ x \end{pmatrix}$ c $U: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x-y \\ x-y \end{pmatrix}$

d $V: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ e $W: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$

4 Find matrix representations for these linear transformations:

a $P: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y+2x \\ -y \end{pmatrix}$ b $Q: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y \\ x+2y \end{pmatrix}$

5 The triangle T has vertices at $(-1, 1)$, $(2, 3)$ and $(5, 1)$.
Find the vertices of the image of T under the transformations represented by these matrices:

a $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ b $\begin{pmatrix} 1 & 4 \\ 0 & -2 \end{pmatrix}$ c $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

6 The square S has vertices at $(-1, 0)$, $(0, 1)$, $(1, 0)$ and $(0, -1)$.
Find the vertices of the image of S under the transformations represented by these matrices:

a $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ b $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ c $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

7 The rectangle R has vertices at $(2, 1)$, $(4, 1)$, $(4, 2)$ and $(2, 2)$.

a Find the vertices of the image of R under the transformation represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

b Sketch R and its image, R', on a coordinate grid.

c Describe fully the transformation that maps R onto R'.

8 A quadrilateral Q has coordinates (1, 0), (4, 2), (3, 4) and (0, 2).

a Find the vertices of the image of Q under the transformation represented by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

b Sketch Q and its image, Q', on a coordinate grid.

c Describe fully the transformation that maps Q onto Q'.

9 A square S has coordinates (−1, 0), (−3, 0), (−3, 2) and (−1, 2).

a Find the vertices of the image of S under the transformation represented by the matrix $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$.

b Sketch S and the image of S on a coordinate grid.

c Describe fully the two transformations that map S onto S'.

10 A triangle T has vertices (4, 1), (4, 3) and (1, 3).

a Find the vertices of the image of T under the transformation represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

b Describe the effect of the transformation represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Exercise B

1 a Write down the matrix representing a reflection in the x-axis.
A triangle has vertices at $A = (1, 3)$, $B = (3, 3)$ and $C = (3, 2)$.

b Use matrices to show that the images of these vertices after a reflection in the x-axis are $A' = (1, -3)$, $B' = (3, -3)$ and $C' = (3, -2)$.

2 a Write down the matrix representing a reflection in the line $y = -x$.
A rectangle has vertices at $P = (1, 1)$, $Q = (1, 3)$, $R = (2, 3)$ and $S = (2, 1)$.

b Use matrices to show that the images of these vertices after a reflection in the line $y = -x$ are $P' = (-1, -1)$, $Q' = (-3, -1)$, $R' = (-3, -2)$ and $S' = (-1, -2)$.

3 Find the matrices that represent the following rotations.

a 90° anticlockwise about (0, 0)

b 270° anticlockwise about (0, 0)

c 45° anticlockwise about (0, 0)

d 210° anticlockwise about (0, 0)

e 135° clockwise about (0, 0)

Watch out The rotation matrix is for angles measured anticlockwise, so make sure you convert the clockwise angle to its equivalent anticlockwise angle.

4 A triangle has vertices at $A = (1, 1)$, $B = (4, 1)$ and $C = (4, 2)$. Find the exact coordinates of the vertices of the triangle after a rotation through:

a 90° anticlockwise about (0, 0)

b 150° anticlockwise about (0, 0)

5 A rectangle has vertices at $P = (2, 2)$, $Q = (2, 3)$, $R = (4, 3)$ and $S = (4, 2)$. Find the exact coordinates of the vertices of the rectangle after a rotation through:

a 270° anticlockwise about (0, 0) **b** 135° clockwise about (0, 0)

6 $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

a Write down fully the transformations represented by the matrices **A** and **B**. **(4 marks)**

b The point (3, 2) is transformed by matrix **A**. Write down the coordinates of the image of this point. **(1 mark)**

c The point (a, b) is transformed onto the point $(a - 3b, 2a - 2b)$ by matrix **B**. Find the values of a and b. **(3 marks)**

7 $\mathbf{M} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

a Write down fully the transformation represented by matrix **M**. **(2 marks)**

b The transformation represented by **M** maps the point (p, q) onto the point C with coordinates $(-\sqrt{2}, -2\sqrt{2})$. Find the values of p and q. **(4 marks)**

c Use your calculator to find $\mathbf{M}^3$. **(1 mark)**

d Point C is mapped onto the point D by the transformation represented by $\mathbf{M}^3$. Find the coordinates of point D and describe fully the transformation represented by $\mathbf{M}^3$. **(2 marks)**

8 a Describe fully the transformation represented by the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. **(2 marks)**

b Write down the equations of three different invariant lines under this transformation. **(2 marks)**

c Write down $\mathbf{A}^{50}$. **(1 mark)**

9 The matrix $\begin{pmatrix} a & b \\ c & -0.5 \end{pmatrix}$ represents an anticlockwise rotation about the origin, through an angle θ.

a Write down the value of a. **(1 mark)**

b Find two possible values of θ, and write down the matrix corresponding to each rotation. **(3 marks)**

10 a Write down the matrix representing a rotation through 270° clockwise about (0, 0). **(1 mark)**

b A point (a, b) transformed using this matrix is such that its image is the point $(a - 5b, 4b)$. Find the values of a and b. **(3 marks)**

Challenge

Prove that the general matrix representing a rotation through angle θ anticlockwise about the origin is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Exercise C

1 Write down the matrices representing the following linear transformations.

 a A stretch with scale factor 4 parallel to the x-axis

 b A stretch with scale factor 3 parallel to the y-axis

 c An enlargement with scale factor 2

 d A stretch with scale factor 5 parallel to the x-axis and a stretch scale factor $\frac{1}{2}$ parallel to the y-axis

2 For each of the transformations in question **1**, write down the area scale factor.

Hint In an enlargement, 'scale factor' refers to the linear scale factor of the enlargement so the area scale factor will be the square of this value.

3 The unit square is transformed using the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$.

 a Write down the coordinates of any invariant points.

 b Work out the area of the resulting rectangle.

4 Write down the matrices representing the following transformations.

 a A stretch with scale factor −2 parallel to the x-axis

 b A stretch with scale factor −3 parallel to the x-axis and scale factor 4 parallel to the y-axis

 c An enlargement with scale factor $-\frac{1}{2}$

5 Prove that the line $y = kx$ is invariant under the transformation $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, where a and k are real constants.

6 $\mathbf{M} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$

 a Describe fully the transformation represented by **M**. **(2 marks)**

 A 2D shape with area k is transformed using the transformation represented by **M**.

 b Given that the image of the shape has area 24, find the value of k. **(2 marks)**

7 A triangle with vertices at (1, 3), (5, 3) and (5, 2) is transformed using the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

 a Find the coordinates of the vertices of the resulting image.

 b Find the area of the new triangle.

8 A rectangle has vertices at (2, 0), (4, 0), (4, 5) and (2, 5). The rectangle is transformed by a stretch with scale factor 2 parallel to the x-axis and a stretch with scale factor −3 parallel to the y-axis.

 a Find the coordinates of vertices of the resulting image.

 b Find the area of the new rectangle.

9 $\mathbf{A} = \begin{pmatrix} 2\sqrt{5} & 0 \\ 0 & 2\sqrt{5} \end{pmatrix}$

 a Describe fully the transformation represented by the matrix **A**. **(2 marks)**

 b A triangle T has coordinates $(a, 1)$, (4, 1) and (4, 3). Given that T is transformed using matrix **A**, and the area of the resulting triangle is 60, find the value of a. **(3 marks)**

10 $\mathbf{M} = \begin{pmatrix} p & 1 \\ p & q \end{pmatrix}$ where p and q are constants and $q > 0$.

a Find $\mathbf{M}^2$ in terms of p and q. **(3 marks)**

Given that $\mathbf{M}^2$ represents an enlargement with centre (0, 0) and scale factor 6,

b find the values of p and q. **(3 marks)**

11 $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & -2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 2 & 5 \end{pmatrix}$

a Find the matrix $\mathbf{M}$ where $\mathbf{M} = \mathbf{AB}$. **(1 mark)**

b Describe fully the transformation represented by the matrix $\mathbf{M}$. **(3 marks)**

c A triangle T has vertices at (2, 1), (6, 1) and (6, k). Given that T is transformed using matrix $\mathbf{M}$, and the resulting triangle has area 320, find the value of k. **(4 marks)**

12 $\mathbf{M} = \begin{pmatrix} -1 & \sqrt{2} \\ -\sqrt{2} & -1 \end{pmatrix}$

A pentagon P of area 12 is transformed using matrix $\mathbf{M}$. Find the area of the image of the pentagon P'. **(2 marks)**

Hint You will learn how to describe transformations such as this in **Section 7.4**.

13 A triangle T has vertices at the points $A = (k, 1)$, $B = (4, 1)$ and $C = (4, k)$ where k is an integer constant.

Triangle T is transformed by the matrix $\begin{pmatrix} 4 & -1 \\ k & 2 \end{pmatrix}$.

Given that triangle T has a right angle at B, and the area of the image triangle T' is 10, find the value of k. **(5 marks)**

14 A triangle T has vertices at (0, 0), (7, 7) and (3, −2).

a Write down the matrix, $\mathbf{M}$, which represents a rotation through 45° anticlockwise about (0, 0). **(1 mark)**

b Find the exact coordinates of the image of T when T is transformed using $\mathbf{M}$. **(3 marks)**

c Show that $\det \mathbf{M} = 1$. **(2 marks)**

d Hence find the area of the original triangle T. **(1 mark)**

Challenge

A transformation U is represented by the matrix $\mathbf{P} = \begin{pmatrix} 7 & 0 \\ 0 & 0 \end{pmatrix}$.

a Find $\det \mathbf{P}$.

b Show that any point in the xy-plane is mapped onto the x-axis by U.

Problem-solving

A non-zero singular 2 × 2 matrix maps any point in the plane onto a straight line through the origin.