Linear Transformations

Important Points:

1.

You can define a transformation in two dimensions by describing how a general point with position vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is transformed. The new point is called the **image**.

2.

- **Linear transformations always map the origin onto itself.**
- **Any linear transformation can be represented by a matrix.**
- **The linear transformation** $T: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$ **can be represented by the matrix** $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ **since** $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$.

Note You can transform any point P by multiplying the transformation matrix by the position vector of P.

3.

Any linear transformation can be defined by the effect it has on unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The transformation represented by the matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ will map $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} b \\ d \end{pmatrix}$.

You can visualise this transformation by considering the unit square:

4.

Points which are mapped onto themselves under the given transformation are called **invariant points**. Lines which map onto themselves are called **invariant lines**.

5.

In reflections, each point on the mirror line is an invariant point.

The mirror line and any lines that are perpendicular the mirror line are invariant lines.

6.

- **The matrix representing a rotation through angle θ anticlockwise about the origin is** $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

The only invariant point is the origin (0, 0). For $\theta \neq 180°$, there are no invariant lines. For $\theta = 180°$, any line passing through the origin is an invariant line.

Note This general rotation matrix is given in the formulae booklet.

7.

- **A transformation represented by the matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is a stretch of scale factor a parallel to the x-axis and a stretch of scale factor b parallel to the y-axis.**

  In the case where $a = b$, the transformation is an enlargement with scale factor a.

Note A stretch parallel to the x-axis only will have matrix $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$.

A stretch parallel to the y-axis only will have matrix $\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$.

- **For any stretch in this form, the x- and y-axes are invariant lines and the origin is an invariant point.**
- **For a stretch parallel to the x-axis only, points on the y-axis are invariant points, and any line parallel to the x-axis is an invariant line.**
- **For a stretch parallel to the y-axis only, points on the x-axis are invariant points, and any line parallel to the y-axis is an invariant line.**

Watch out For a stretch in both directions, although the axes are invariant lines, the points on the axes are not themselves invariant. Any point on an axis (apart from the origin) will map to a **different** point on the same axis.

Reflections and rotations of 2D shapes both preserve the area of a shape. When a shape is stretched, its area can increase or decrease. You can use the determinant of the matrix representing this transformation to work out the scale factor for the change in area.

- **For a linear transformation represented by matrix M, det M represents the scale factor for the change in area. This is sometimes called the area scale factor.**

Watch out If the determinant of the matrix **M** is negative, the shape has been reflected.