Linear Transformations

Exercise A

1 Which of the following transformations are linear transformations?

a $P: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x \\ y+1 \end{pmatrix}$ **b** $Q: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ y \end{pmatrix}$ **c** $R: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x+y \\ x+xy \end{pmatrix}$

d $S: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -x \end{pmatrix}$ **e** $T: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y+3 \\ x+3 \end{pmatrix}$ **f** $U: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x \\ 3y-2x \end{pmatrix}$

2 Identify which of these are linear transformations and give their matrix representations. Give reasons to explain why the other transformations are not linear.

a $S: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x-y \\ 3x \end{pmatrix}$ **b** $T: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2y+1 \\ x-1 \end{pmatrix}$ **c** $U: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} xy \\ 0 \end{pmatrix}$

d $V: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2y \\ -x \end{pmatrix}$ **e** $W: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}$

3 Identify which of these are linear transformations and give their matrix representations. Give reasons to explain why the other transformations are not linear.

a $S: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$ **b** $T: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y \\ x \end{pmatrix}$ **c** $U: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x-y \\ x-y \end{pmatrix}$

d $V: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ **e** $W: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$

4 Find matrix representations for these linear transformations:

a $P: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y+2x \\ -y \end{pmatrix}$ **b** $Q: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y \\ x+2y \end{pmatrix}$

5 The triangle T has vertices at $(-1, 1)$, $(2, 3)$ and $(5, 1)$.
Find the vertices of the image of T under the transformations represented by these matrices:

a $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 4 \\ 0 & -2 \end{pmatrix}$ **c** $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

6 The square S has vertices at $(-1, 0)$, $(0, 1)$, $(1, 0)$ and $(0, -1)$.
Find the vertices of the image of S under the transformations represented by these matrices:

a $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

7 The rectangle R has vertices at $(2, 1)$, $(4, 1)$, $(4, 2)$ and $(2, 2)$.

a Find the vertices of the image of R under the transformation represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

b Sketch R and its image, R', on a coordinate grid.

c Describe fully the transformation that maps R onto R'.

8 A quadrilateral Q has coordinates (1, 0), (4, 2), (3, 4) and (0, 2).

a Find the vertices of the image of Q under the transformation represented by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

b Sketch Q and its image, Q', on a coordinate grid.

c Describe fully the transformation that maps Q onto Q'.

9 A square S has coordinates (−1, 0), (−3, 0), (−3, 2) and (−1, 2).

a Find the vertices of the image of S under the transformation represented by the matrix $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$.

b Sketch S and the image of S on a coordinate grid.

c Describe fully the two transformations that map S onto S'.

10 A triangle T has vertices (4, 1), (4, 3) and (1, 3).

a Find the vertices of the image of T under the transformation represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

b Describe the effect of the transformation represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Exercise B

1 a Write down the matrix representing a reflection in the x-axis.
A triangle has vertices at $A = (1, 3)$, $B = (3, 3)$ and $C = (3, 2)$.

b Use matrices to show that the images of these vertices after a reflection in the x-axis are $A' = (1, -3)$, $B' = (3, -3)$ and $C' = (3, -2)$.

2 a Write down the matrix representing a reflection in the line $y = -x$.
A rectangle has vertices at $P = (1, 1)$, $Q = (1, 3)$, $R = (2, 3)$ and $S = (2, 1)$.

b Use matrices to show that the images of these vertices after a reflection in the line $y = -x$ are $P' = (-1, -1)$, $Q' = (-3, -1)$, $R' = (-3, -2)$ and $S' = (-1, -2)$.

3 Find the matrices that represent the following rotations.

a 90° anticlockwise about (0, 0)

b 270° anticlockwise about (0, 0)

c 45° anticlockwise about (0, 0)

d 210° anticlockwise about (0, 0)

e 135° clockwise about (0, 0)

Watch out The rotation matrix is for angles measured anticlockwise, so make sure you convert the clockwise angle to its equivalent anticlockwise angle.

4 A triangle has vertices at $A = (1, 1)$, $B = (4, 1)$ and $C = (4, 2)$. Find the exact coordinates of the vertices of the triangle after a rotation through:

a 90° anticlockwise about (0, 0) **b** 150° anticlockwise about (0, 0)

5 A rectangle has vertices at $P = (2, 2)$, $Q = (2, 3)$, $R = (4, 3)$ and $S = (4, 2)$. Find the exact coordinates of the vertices of the rectangle after a rotation through:

a 270° anticlockwise about (0, 0) **b** 135° clockwise about (0, 0)

6 $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

a Write down fully the transformations represented by the matrices **A** and **B**. **(4 marks)**

b The point (3, 2) is transformed by matrix **A**. Write down the coordinates of the image of this point. **(1 mark)**

c The point (a, b) is transformed onto the point $(a - 3b, 2a - 2b)$ by matrix **B**. Find the values of a and b. **(3 marks)**

7 $\mathbf{M} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

a Write down fully the transformation represented by matrix **M**. **(2 marks)**

b The transformation represented by **M** maps the point (p, q) onto the point C with coordinates $(-\sqrt{2}, -2\sqrt{2})$. Find the values of p and q. **(4 marks)**

c Use your calculator to find $\mathbf{M}^3$. **(1 mark)**

d Point C is mapped onto the point D by the transformation represented by $\mathbf{M}^3$. Find the coordinates of point D and describe fully the transformation represented by $\mathbf{M}^3$. **(2 marks)**

8 **a** Describe fully the transformation represented by the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. **(2 marks)**

b Write down the equations of three different invariant lines under this transformation. **(2 marks)**

c Write down $\mathbf{A}^{50}$. **(1 mark)**

9 The matrix $\begin{pmatrix} a & b \\ c & -0.5 \end{pmatrix}$ represents an anticlockwise rotation about the origin, through an angle θ.

a Write down the value of a. **(1 mark)**

b Find two possible values of θ, and write down the matrix corresponding to each rotation. **(3 marks)**

10 **a** Write down the matrix representing a rotation through 270° clockwise about (0, 0). **(1 mark)**

b A point (a, b) transformed using this matrix is such that its image is the point $(a - 5b, 4b)$. Find the values of a and b. **(3 marks)**

Challenge

Prove that the general matrix representing a rotation through angle θ anticlockwise about the origin is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.