Complex Numbers 1

Exercise A

Do not use your calculator in this exercise.

1 Write each of the following in the form bi where b is a real number.

a $\sqrt{-9}$ **b** $\sqrt{-49}$ **c** $\sqrt{-121}$ **d** $\sqrt{-10\,000}$ **e** $\sqrt{-225}$

f $\sqrt{-5}$ **g** $\sqrt{-12}$ **h** $\sqrt{-45}$ **i** $\sqrt{-200}$ **j** $\sqrt{-147}$

2 Simplify, giving your answers in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$.

a $(5 + 2\text{i}) + (8 + 9\text{i})$ **b** $(4 + 10\text{i}) + (1 - 8\text{i})$

c $(7 + 6\text{i}) + (-3 - 5\text{i})$ **d** $\left(\frac{1}{2} + \frac{1}{3}\text{i}\right) + \left(\frac{5}{2} + \frac{5}{3}\text{i}\right)$

e $(20 + 12\text{i}) - (11 + 3\text{i})$ **f** $(2 - \text{i}) - (-5 + 3\text{i})$

g $(-4 - 6\text{i}) - (-8 - 8\text{i})$ **h** $(3\sqrt{2} + \text{i}) - (\sqrt{2} - \text{i})$

i $(-2 - 7\text{i}) + (1 + 3\text{i}) - (-12 + \text{i})$ **j** $(18 + 5\text{i}) - (15 - 2\text{i}) - (3 + 7\text{i})$

3 Simplify, giving your answers in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$.

a $2(7 + 2\text{i})$ **b** $3(8 - 4\text{i})$

c $2(3 + \text{i}) + 3(2 + \text{i})$ **d** $5(4 + 3\text{i}) - 4(-1 + 2\text{i})$

e $\dfrac{6 - 4\text{i}}{2}$ **f** $\dfrac{15 + 25\text{i}}{5}$

g $\dfrac{9 + 11\text{i}}{3}$ **h** $\dfrac{-8 + 3\text{i}}{4} - \dfrac{7 - 2\text{i}}{2}$

4 Write in the form $a + b\text{i}$, where a and b are simplified surds.

a $\dfrac{4 - 2\text{i}}{\sqrt{2}}$ **b** $\dfrac{2 - 6\text{i}}{1 + \sqrt{3}}$

5 Given that $z = 7 - 6\text{i}$ and $w = 7 + 6\text{i}$, find, in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$:

a $z - w$ **b** $w + z$

Notation Complex numbers are often represented by the letter z or the letter w.

6 Given that $z_1 = a + 9\text{i}$, $z_2 = -3 + b\text{i}$ and $z_2 - z_1 = 7 + 2\text{i}$, find a and b where $a, b \in \mathbb{R}$. **(2 marks)**

7 Given that $z_1 = 4 + \text{i}$ and $z_2 = 7 - 3\text{i}$, find, in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$:

a $z_1 - z_2$ **b** $4z_2$ **c** $2z_1 + 5z_2$

8 Given that $z = a + b\text{i}$ and $w = a - b\text{i}$, $a, b \in \mathbb{R}$, show that:

a $z + w$ is always real **b** $z - w$ is always imaginary

Exercise B

1 Solve each of the following equations. Write your answers in the form $\pm b\text{i}$.

a $z^2 + 121 = 0$ **b** $z^2 + 40 = 0$ **c** $2z^2 + 120 = 0$

d $3z^2 + 150 = 38 - z^2$ **e** $z^2 + 30 = -3z^2 - 66$ **f** $6z^2 + 1 = 2z^2$

2 Solve each of the following equations. Write your answers in the form $a \pm b\text{i}$.

a $(z - 3)^2 - 9 = -16$

b $2(z - 7)^2 + 30 = 6$

c $16(z + 1)^2 + 11 = 2$

Hint The left-hand side of each equation is in completed square form already. Use inverse operations to find the values of z.

3 Solve each of the following equations. Write your answers in the form $a \pm bi$.

a $z^2 + 2z + 5 = 0$ **b** $z^2 - 2z + 10 = 0$ **c** $z^2 + 4z + 29 = 0$

d $z^2 + 10z + 26 = 0$ **e** $z^2 + 5z + 25 = 0$ **f** $z^2 + 3z + 5 = 0$

4 Solve each of the following equations. Write your answers in the form $a \pm bi$.

a $2z^2 + 5z + 4 = 0$ **b** $7z^2 - 3z + 3 = 0$ **c** $5z^2 - z + 3 = 0$

5 The solutions to the quadratic equation $z^2 - 8z + 21 = 0$ are z_1 and z_2.
Find z_1 and z_2, giving each in the form $a \pm i\sqrt{b}$.

6 The equation $z^2 + bz + 11 = 0$, where $b \in \mathbb{R}$, has distinct non-real complex roots.
Find the range of possible values of b. **(3 marks)**

Exercise C

Do not use your calculator in this exercise.

1 Simplify each of the following, giving your answers in the form $a + b\text{i}$.

a $(5 + \text{i})(3 + 4\text{i})$ **b** $(6 + 3\text{i})(7 + 2\text{i})$ **c** $(5 - 2\text{i})(1 + 5\text{i})$

d $(13 - 3\text{i})(2 - 8\text{i})$ **e** $(-3 - \text{i})(4 + 7\text{i})$ **f** $(8 + 5\text{i})^2$

g $(2 - 9\text{i})^2$ **h** $(1 + \text{i})(2 + \text{i})(3 + \text{i})$

i $(3 - 2\text{i})(5 + \text{i})(4 - 2\text{i})$ **j** $(2 + 3\text{i})^3$

Hint For part **h**, begin by multiplying the first pair of brackets.

2 **a** Simplify $(4 + 5\text{i})(4 - 5\text{i})$, giving your answer in the form $a + b\text{i}$.

b Simplify $(7 - 2\text{i})(7 + 2\text{i})$, giving your answer in the form $a + b\text{i}$.

c Comment on your answers to parts **a** and **b**.

d Prove that $(a + b\text{i})(a - b\text{i})$ is a real number for any real numbers a and b.

3 Given that $(a + 3\text{i})(1 + b\text{i}) = 25 - 39\text{i}$, find two possible pairs of values for a and b.

4 Write each of the following in its simplest form.

a i^6 **b** $(3\text{i})^4$ **c** $\text{i}^5 + \text{i}$ **d** $(4\text{i})^3 - 4\text{i}^3$

5 Express $(1 + \text{i})^6$ in the form $a - b\text{i}$, where a and b are integers to be found.

6 Find the value of the real part of $(3 - 2\text{i})^4$.

Problem-solving

You can use the binomial theorem to expand $(a + b)^n$. ← Pure Year 1, Section 8.3

7 $f(z) = 2z^2 - z + 8$

Find: **a** $f(2\text{i})$ **b** $f(3 - 6\text{i})$

8 $f(z) = z^2 - 2z + 17$

Show that $z = 1 - 4\text{i}$ is a solution to $f(z) = 0$. **(2 marks)**

9 **a** Given that $\text{i}^1 = \text{i}$ and $\text{i}^2 = -1$, write i^3 and i^4 in their simplest forms.

b Write i^5, i^6, i^7 and i^8 in their simplest forms.

c Write down the value of:

i i^{100} **ii** i^{253} **iii** i^{301}

Exercise D

Do not use your calculator in this exercise.

1 Write down the complex conjugate z^* for:

a $z = 8 + 2i$ b $z = 6 - 5i$ c $z = \frac{2}{3} - \frac{1}{2}i$ d $z = \sqrt{5} + i\sqrt{10}$

2 Find $z + z^*$ and zz^* for:

a $z = 6 - 3i$ b $z = 10 + 5i$ c $z = \frac{3}{4} + \frac{1}{4}i$ d $z = \sqrt{5} - 3i\sqrt{5}$

3 Write each of the following in the form $a + bi$.

a $\frac{3 - 5i}{1 + 3i}$ b $\frac{3 + 5i}{6 - 8i}$ c $\frac{28 - 3i}{1 - i}$ d $\frac{2 + i}{1 + 4i}$

4 Write $\frac{(3 - 4i)^2}{1 + i}$ in the form $x + iy$ where $x, y \in \mathbb{R}$.

5 Given that $z_1 = 1 + i$, $z_2 = 2 + i$ and $z_3 = 3 + i$, write each of the following in the form $a + bi$.

a $\frac{z_1 z_2}{z_3}$ b $\frac{(z_2)^2}{z_1}$ c $\frac{2z_1 + 5z_3}{z_2}$

6 Given that $\frac{5 + 2i}{z} = 2 - i$, find z in the form $a + bi$. **(2 marks)**

7 Simplify $\frac{6 + 8i}{1 + i} + \frac{6 + 8i}{1 - i}$, giving your answer in the form $a + bi$.

8 $w = \frac{4}{8 - i\sqrt{2}}$

Express w in the form $a + bi\sqrt{2}$, where a and b are rational numbers.

9 $w = 1 - 9i$

Express $\frac{1}{w}$ in the form $a + bi$, where a and b are rational numbers.

10 $z = 4 - i\sqrt{2}$

Use algebra to express $\frac{z + 4}{z - 3}$ in the form $p + qi\sqrt{2}$, where p and q are rational numbers.

11 The complex number z satisfies the equation $(4 + 2i)(z - 2i) = 6 - 4i$.

Find z, giving your answer in the form $a + bi$ where a and b are rational numbers. **(4 marks)**

12 The complex numbers z_1 and z_2 are given by $z_1 = p - 7i$ and $z_2 = 2 + 5i$ where p is an integer.

Find $\frac{z_1}{z_2}$ in the form $a + bi$ where a and b are rational, and are given in terms of p. **(4 marks)**

13 $z = \sqrt{5} + 4i$. z^* is the complex conjugate of z.

Show that $\frac{z}{z^*} = a + bi\sqrt{5}$, where a and b are rational numbers to be found. **(4 marks)**

14 The complex number z is defined by $z = \frac{p + 5i}{p - 2i}$, $p \in \mathbb{R}, p > 0$.

Given that the real part of z is $\frac{1}{2}$,

a find the value of p **(4 marks)**

b write z in the form $a + bi$, where a and b are real. **(1 mark)**

Exercise E

1 The roots of the quadratic equation $z^2 + 2z + 26 = 0$ are α and β.

Find: **a** α and β **b** $\alpha + \beta$ **c** $\alpha\beta$

2 The roots of the quadratic equation $z^2 - 8z + 25 = 0$ are α and β.

Find: **a** α and β **b** $\alpha + \beta$ **c** $\alpha\beta$

3 Given that $2 + 3i$ is one of the roots of a quadratic equation with real coefficients,

a write down the other root of the equation **(1 mark)**

b find the quadratic equation, giving your answer in the form $z^2 + bz + c = 0$ where b and c are real constants. **(3 marks)**

4 Given that $5 - i$ is a root of the equation $z^2 + pz + q = 0$, where p and q are real constants,

a write down the other root of the equation **(1 mark)**

b find the value of p and the value of q. **(3 marks)**

5 Given that $z_1 = -5 + 4i$ is one of the roots of the quadratic equation $z^2 + bz + c = 0$, where b and c are real constants, find the values of b and c. **(4 marks)**

6 Given that $1 + 2i$ is one of the roots of a quadratic equation with real coefficients, find the equation giving your answer in the form $z^2 + bz + c = 0$ where b and c are integers to be found. **(4 marks)**

7 Given that $3 - 5i$ is one of the roots of a quadratic equation with real coefficients, find the equation giving your answer in the form $z^2 + bz + c = 0$ where b and c are real constants. **(4 marks)**

8 $z = \dfrac{5}{3 - i}$

a Find z in the form $a + bi$, where a and b are real constants. **(1 mark)**

Given that z is a complex root of the quadratic equation $x^2 + px + q = 0$, where p and q are real integers,

b find the value of p and the value of q. **(4 marks)**

9 Given that $z = 5 + qi$ is a root of the equation $z^2 - 4pz + 34 = 0$, where p and q are positive real constants, find the value of p and the value of q. **(4 marks)**

Exercise F

1 $f(z) = z^3 - 6z^2 + 21z - 26$

a Show that $f(2) = 0$. **(1 mark)**

b Hence solve $f(z) = 0$ completely. **(3 marks)**

2 $f(z) = 2z^3 + 5z^2 + 9z - 6$

a Show that $f(\frac{1}{2}) = 0$. **(1 mark)**

b Hence write $f(z)$ in the form $(2z - 1)(z^2 + bz + c)$, where b and c are real constants to be found. **(2 marks)**

c Use algebra to solve $f(z) = 0$ completely. **(2 marks)**

3 $g(z) = 2z^3 - 4z^2 - 5z - 3$

Given that $z = 3$ is a root of the equation $g(z) = 0$, solve $g(z) = 0$ completely. **(4 marks)**

4 $p(z) = z^3 + 4z^2 - 15z - 68$

Given that $z = -4 + i$ is a solution to the equation $p(z) = 0$,

a show that $z^2 + 8z + 17$ is a factor of $p(z)$. **(2 marks)**

b Hence solve $p(z) = 0$ completely. **(2 marks)**

5 $f(z) = z^3 + 9z^2 + 33z + 25$

Given that $f(z) = (z + 1)(z^2 + az + b)$, where a and b are real constants,

a find the value of a and the value of b **(2 marks)**

b find the three roots of $f(z) = 0$ **(4 marks)**

c find the sum of the three roots of $f(z) = 0$. **(1 mark)**

6 $g(z) = z^3 - 12z^2 + cz + d = 0$, where $c, d \in \mathbb{R}$.

Given that 6 and $3 + i$ are roots of the equation $g(z) = 0$,

a write down the other complex root of the equation **(1 mark)**

b find the value of c and the value of d. **(4 marks)**

7 $h(z) = 2z^3 + 3z^2 + 3z + 1$

Given that $2z + 1$ is a factor of $h(z)$, find the three roots of $h(z) = 0$. **(4 marks)**

8 $f(z) = z^3 - 6z^2 + 28z + k$

Given that $f(2) = 0$,

a find the value of k **(1 mark)**

b find the other two roots of the equation. **(4 marks)**

9 Find the four roots of the equation $z^4 - 16 = 0$.

10 $f(z) = z^4 - 12z^3 + 31z^2 + 108z - 360$

a Write $f(z)$ in the form $(z^2 - 9)(z^2 + bz + c)$, where b and c are real constants to be found. **(2 marks)**

b Hence find all the solutions to $f(z) = 0$. **(3 marks)**

11 $g(z) = z^4 + 2z^3 - z^2 + 38z + 130$

Given that $g(2 + 3i) = 0$, find all the roots of $g(z) = 0$.

12 $f(z) = z^4 - 10z^3 + 71z^2 + Qz + 442$, where Q is a real constant.

Given that $z = 2 - 3i$ is a root of the equation $f(z) = 0$,

a show that $z^2 - 6z + 34$ is a factor of $f(z)$ **(4 marks)**

b find the value of Q **(1 mark)**

c solve completely the equation $f(z) = 0$. **(2 marks)**

Exercise G

1 Given that $z_1 = 8 - 3i$ and $z_2 = -2 + 4i$, find, in the form $a + bi$, where $a, b \in \mathbb{R}$:

a $z_1 + z_2$

b $3z_2$

c $6z_1 - z_2$

2 The equation $z^2 + bz + 14 = 0$, where $b \in \mathbb{R}$ has no real roots.

Find the range of possible values of b. **(3 marks)**

3 The solutions to the quadratic equation $z^2 - 6z + 12 = 0$ are z_1 and z_2.

Find z_1 and z_2, giving each answer in the form $a \pm i\sqrt{b}$.

4 By using the binomial expansion, or otherwise, show that $(1 + 2\text{i})^5 = 41 - 38\text{i}$. **(3 marks)**

5 $\text{f}(z) = z^2 - 6z + 10$

Show that $z = 3 + \text{i}$ is a solution to $\text{f}(z) = 0$. **(2 marks)**

6 $z_1 = 4 + 2\text{i}$, $z_2 = -3 + \text{i}$

Express, in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$:

a z_1^* **b** $z_1 z_2$ **c** $\dfrac{z_1}{z_2}$

7 Write $\dfrac{(7 - 2\text{i})^2}{1 + \text{i}\sqrt{3}}$ in the form $x + \text{i}y$ where $x, y \in \mathbb{R}$.

8 Given that $\dfrac{4 - 7\text{i}}{z} = 3 + \text{i}$, find z in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$. **(2 marks)**

9 $z = \dfrac{1}{2 + \text{i}}$

Express in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$:

a z^2 **b** $z - \dfrac{1}{z}$

10 Given that $z = a + b\text{i}$, show that $\dfrac{z}{z^*} = \left(\dfrac{a^2 - b^2}{a^2 + b^2}\right) + \left(\dfrac{2ab}{a^2 + b^2}\right)\text{i}$ **(4 marks)**

11 The complex number z is defined by $z = \dfrac{3 + q\text{i}}{q - 5\text{i}}$, where $q \in \mathbb{R}$.

Given that the real part of z is $\frac{1}{13}$,

a find the possible values of q **(4 marks)**

b write the possible values of z in the form $a + b\text{i}$, where a and b are real constants. **(1 mark)**

12 Given that $z = x + \text{i}y$, find the value of x and the value of y such that $z + 4\text{i}z^* = -3 + 18\text{i}$ where z^* is the complex conjugate of z. **(5 marks)**

13 $z = 9 + 6\text{i}$, $w = 2 - 3\text{i}$

Express $\frac{z}{w}$ in the form $a + b\text{i}$, where a and b are real constants.

14 The complex number z is given by $z = \dfrac{q + 3\text{i}}{4 + q\text{i}}$ where q is an integer.

Express z in the form $a + b\text{i}$ where a and b are rational and are given in terms of q. **(4 marks)**

15 Given that $6 - 2\text{i}$ is one of the roots of a quadratic equation with real coefficients,

a write down the other root of the equation **(1 mark)**

b find the quadratic equation, giving your answer in the form $z^2 + bz + c = 0$ where b and c are real constants. **(2 marks)**

16 Given that $z = 4 - k\text{i}$ is a root of the equation $z^2 - 2mz + 52 = 0$, where k and m are positive real constants, find the value of k and the value of m. **(4 marks)**

17 $\text{h}(z) = z^3 - 11z + 20$

Given that $2 + \text{i}$ is a root of the equation $\text{h}(z) = 0$, solve $\text{h}(z) = 0$ completely. **(4 marks)**

18 $\text{f}(z) = z^3 + 6z + 20$

Given that $\text{f}(1 + 3\text{i}) = 0$, solve $\text{f}(z) = 0$ completely. **(4 marks)**

19 $\text{f}(z) = z^3 + 3z^2 + kz + 48$, $k \in \mathbb{R}$

Given that $\text{f}(4\text{i}) = 0$,

a find the value of k **(2 marks)**

b find the other two roots of the equation. **(3 marks)**

20 $f(z) = z^4 - z^3 - 16z^2 - 74z - 60$

a Write $f(z)$ in the form $(z^2 - 5z - 6)(z^2 + bz + c)$, where b and c are real constants to be found. **(2 marks)**

b Hence find all the solutions to $f(z) = 0$. **(3 marks)**

21 $g(z) = z^4 - 6z^3 + 19z^2 - 36z + 78$

Given that $g(3 - 2i) = 0$, find all the roots of $g(z) = 0$. **(4 marks)**

22 $f(z) = z^4 - 2z^3 - 5z^2 + pz + 24$

Given that $f(4) = 0$,

a find the value of p **(1 mark)**

b solve completely the equation $f(z) = 0$. **(5 marks)**