Matrices 1

Exercise A

1 Write the size of each matrix in the form $n \times m$.

a $\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & -1 \end{pmatrix}$

d $(1 \quad 2 \quad 3)$ **e** $(3 \quad -1)$ **f** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2 Write down the 4×4 identity matrix, $\mathbf{I}_4$.

3 Two matrices **A** and **B** are given as:

$\mathbf{A} = \begin{pmatrix} 1 & 3 & a \\ 2 & -1 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 3 & 6 \\ b & -1 & 4 \end{pmatrix}$

If $\mathbf{A} = \mathbf{B}$, write down the values of a and b.

4 For the matrices

$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ -1 & -2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$

find:

a $\mathbf{A} + \mathbf{C}$ **b** $\mathbf{B} - \mathbf{A}$ **c** $\mathbf{A} + \mathbf{B} - \mathbf{C}$

5 For the matrices

$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{B} = (1 \quad -1)$, $\mathbf{C} = (-1 \quad 1 \quad 0)$,

$\mathbf{D} = (0 \quad 1 \quad -1)$, $\mathbf{E} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\mathbf{F} = (2 \quad 1 \quad 3)$

find where possible:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} - \mathbf{E}$ **c** $\mathbf{F} - \mathbf{D} + \mathbf{C}$ **d** $\mathbf{B} + \mathbf{C}$

e $\mathbf{F} - (\mathbf{D} + \mathbf{C})$ **f** $\mathbf{A} - \mathbf{F}$ **g** $\mathbf{C} - (\mathbf{F} - \mathbf{D})$

6 Given that $\begin{pmatrix} a & 2 \\ -1 & b \end{pmatrix} - \begin{pmatrix} 1 & c \\ d & -2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$, find the values of the constants a, b, c and d.

(P) **7** Given that $\begin{pmatrix} 1 & 2 & 0 \\ a & b & c \end{pmatrix} + \begin{pmatrix} a & b & c \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} c & 5 & c \\ c & c & c \end{pmatrix}$, find the values of a, b and c.

8 Given that $\begin{pmatrix} 5 & 3 \\ 0 & -1 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ 2 & 0 \\ 1 & 4 \end{pmatrix}$, find the values of a, b, c, d, e and f.

9 For the matrices $\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 2 \\ 3 & 4 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 & 3 & -4 \\ 1 & 1 & 2 \\ -2 & 0 & -3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -2 & 0 & 3 \\ 2 & 8 & -6 \\ -1 & 1 & 1 \end{pmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{B} - \mathbf{C}$ **c** $\mathbf{C} + \mathbf{A}$

d A matrix $\mathbf{M} = \begin{pmatrix} 5 & -6 & b \\ 4 & a & 6 \\ 2 & 0 & c \end{pmatrix}$. Find the values of a, b and c if:

i $\mathbf{A} + \mathbf{M} = \begin{pmatrix} 6 & -7 & -2 \\ 6 & 3 & 8 \\ 5 & 4 & 6 \end{pmatrix}$ **ii** $\mathbf{M} - \mathbf{B} = \begin{pmatrix} -1 & -9 & -5 \\ -2 & 7 & -1 \\ -3 & 2 & 2 \end{pmatrix}$

11 The matrices **A** and **B** are defined as:

$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}$

Find:

a $3\mathbf{A} + 2\mathbf{B}$ **b** $2\mathbf{A} - 4\mathbf{B}$ **c** $5\mathbf{A} - 2\mathbf{B}$ **d** $\frac{1}{2}\mathbf{A} + \frac{3}{2}\mathbf{B}$

12 The matrices **M** and **N** are defined as:

$\mathbf{M} = \begin{pmatrix} 2 & 4 & -1 \\ 1 & -3 & -1 \\ 0 & 2 & 2 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 6 & -2 & 5 \\ 3 & -3 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

Find:

a $\mathbf{M} + 2\mathbf{N}$ **b** $3\mathbf{M} - \mathbf{N}$ **c** $4\mathbf{M} + 5\mathbf{N}$ **d** $\frac{2}{3}\mathbf{M} - \frac{1}{2}\mathbf{N}$

13 Find the value of k and the value of x such that $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} + k\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 7 \\ x & 0 \end{pmatrix}$.

14 Find the values of a, b, c and d such that $2\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} - 3\begin{pmatrix} 1 & c \\ d & -1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -4 & -4 \end{pmatrix}$.

15 Find the values of a, b, c and d such that $\begin{pmatrix} 5 & a \\ b & 0 \end{pmatrix} - 2\begin{pmatrix} c & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 1 \\ 3 & d \end{pmatrix}$.

(P) **16** Find the value of k such that $\begin{pmatrix} -3 \\ k \end{pmatrix} + k\begin{pmatrix} 2k \\ 2k \end{pmatrix} = \begin{pmatrix} k \\ 6 \end{pmatrix}$.

(P) **17** The matrices **A** and **B** are defined as:

$\mathbf{A} = \begin{pmatrix} p & 0 & 0 \\ 0 & q^2 & r \\ 0 & 0 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2q & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{pmatrix}$

where p, q and r are positive constants.

Given that $\mathbf{A} - k\mathbf{B} = \mathbf{I}_3$, where $\mathbf{I}_3$ is the 3×3 identity matrix, find:

a the value of k **b** the values of p, q and r.

(P) **18** The matrices **P** and **Q** are defined as:

$\mathbf{P} = \begin{pmatrix} 0 & 2 & c \\ a & 0 & 0 \\ 0 & \mathrm{b} & -1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 0 & -1 & -1 \\ 3 & d & 0 \\ 0 & 2 & e \end{pmatrix}$

where a, b, c, d, and e are constants.

Given that $\mathbf{P} - k\mathbf{Q} = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix, find the values of a, b, c, d, e and k.

Exercise B

1 Given the sizes of the following matrices:

Matrix	A	B	C	D	E
Size	2×2	1×2	1×3	3×2	2×3

find the sizes of these matrix products.

a **BA** **b** **DE** **c** **CD**

d **ED** **e** **AE** **f** **DA**

2 Use your calculator to find these products:

a $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 5 \\ -1 & -2 \end{pmatrix}$

3 The matrix $\mathbf{A} = \begin{pmatrix} -1 & -2 \\ 0 & 3 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Use your calculator to find:

a $\mathbf{AB}$ **b** $\mathbf{A}^2$

Hint $\mathbf{A}^2$ means $\mathbf{A} \times \mathbf{A}$.

4 The matrices **A**, **B** and **C** are given by:

$\mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -3 & -2 \end{pmatrix}$

Without using your calculator, determine whether or not the following products exist and fin the products of those that do.

a $\mathbf{AB}$ **b** $\mathbf{AC}$ **c** $\mathbf{BC}$

d $\mathbf{BA}$ **e** $\mathbf{CA}$ **f** $\mathbf{CB}$

5 Find $\begin{pmatrix} 2 & a \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 2 \end{pmatrix}$, giving your answer in terms of a.

6 Find $\begin{pmatrix} 3 & 2 \\ -1 & x \end{pmatrix}\begin{pmatrix} x & -2 \\ 1 & 3 \end{pmatrix}$, giving your answer in terms of x.

7 The matrices **A**, **B** and **C** are defined as:

$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -3 & 1 \\ 1 & 2 \end{pmatrix}$.

Use your calculator to find:

a $\mathbf{AB} - \mathbf{C}$ **b** $\mathbf{BC} + 3\mathbf{A}$ **c** $4\mathbf{B} - 3\mathbf{CA}$

8 The matrices **M** and **N** are defined as:

$\mathbf{M} = \begin{pmatrix} 3 & k \\ k & 1 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix}$. Find, in terms of k:

a $\mathbf{MN}$ **b** $\mathbf{NM}$ **c** $3\mathbf{M} - 2\mathbf{N}$ **d** $2\mathbf{MN} + 3\mathbf{N}$

9 The matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Find:

a $\mathbf{A}^2$

b $\mathbf{A}^3$

c Suggest a form for $\mathbf{A}^k$.

Links You might be asked to prove this general form for $\mathbf{A}^k$. → **Section 8.3**

10 The matrix $\mathbf{A} = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$.

a Find, in terms of a and b, the matrix $\mathbf{A}^2$.

Given that $\mathbf{A}^2 = 3\mathbf{A}$,

b find the value of a.

11 $\mathbf{A} = \begin{pmatrix} -1 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 4 & -2 \\ 0 & -3 \end{pmatrix}$

Find: **a** $\mathbf{BAC}$ **b** $\mathbf{AC}^2$

12 $\mathbf{A} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{B} = (3 \quad -2 \quad -3)$

Find: **a** **ABA** **b** **BAB**

13 **a** Write down $\mathbf{I}_2$.

b Given that matrix $\mathbf{A} = \begin{pmatrix} 2 & -2 \\ 1 & 3 \end{pmatrix}$, show that $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$.

14 $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 & 2 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$.

Show that $\mathbf{AB} + \mathbf{AC} = \mathbf{A}(\mathbf{B} + \mathbf{C})$.

15 $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{I}$ is the 2×2 identity matrix.

Prove that $\mathbf{A}^2 = 2\mathbf{A} + 5\mathbf{I}$. **(2 marks)**

16 A matrix **M** is given as $\mathbf{M} = \begin{pmatrix} 1 & 2 & c \\ a & -1 & 1 \\ 1 & b & 0 \end{pmatrix}$.

Find $\mathbf{M}^2$ in terms of a, b and c. **(3 marks)**

17 A matrix **A** is given as $\mathbf{A} = \begin{pmatrix} 1 & -1 & b \\ a & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$.

Given that $\mathbf{A}^2 = \begin{pmatrix} -4 & -3 & -8 \\ 9 & 1 & -6 \\ 4 & -1 & 7 \end{pmatrix}$, find the values of a and b. **(3 marks)**

18 $\mathbf{A} = \begin{pmatrix} p & 3 \\ 6 & p \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} q & 2 \\ 4 & q \end{pmatrix}$, where p and q are constants. Prove that $\mathbf{AB} = \mathbf{BA}$. **(3 marks)**

19 The matrix $\mathbf{A} = \begin{pmatrix} 3 & p \\ -4 & q \end{pmatrix}$ is such that $\mathbf{A}^2 = \mathbf{I}$. Find the values of p and q. **(3 marks)**

Exercise C

1 Find the determinants of the following matrices.

a $\begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix}$ **c** $\begin{pmatrix} -2 & 1 \\ 3 & 0 \end{pmatrix}$

d $\begin{pmatrix} -4 & -4 \\ 1 & 1 \end{pmatrix}$ **e** $\begin{pmatrix} 7 & -4 \\ 0 & 3 \end{pmatrix}$ **f** $\begin{pmatrix} -1 & -1 \\ -6 & -10 \end{pmatrix}$

2 Find the values of a for which these matrices are singular.

a $\begin{pmatrix} a & 1+a \\ 3 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1+a & 3-a \\ a+2 & 1-a \end{pmatrix}$ **c** $\begin{pmatrix} 2+a & 1-a \\ 1-a & a \end{pmatrix}$

3 Given that k is a real number and that $\mathbf{M} = \begin{pmatrix} -2 & 1-k \\ k-1 & k \end{pmatrix}$,

find the exact values of k for which $\mathbf{M}$ is a singular matrix. **(3 marks)**

4 $\mathbf{P} = \begin{pmatrix} 3k & 4-k \\ k-2 & -k \end{pmatrix}$, where k is a real constant.

Given that $\mathbf{P}$ is a singular matrix, find the possible values of k. **(3 marks)**

5 The matrix $\mathbf{A} = \begin{pmatrix} a & 2a \\ b & 2b \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 2b & -2a \\ -b & a \end{pmatrix}$.

a Find det $\mathbf{A}$ and det $\mathbf{B}$.

b Find $\mathbf{AB}$.

6 Use your calculator to find the values of these determinants.

a $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$ **b** $\begin{vmatrix} 0 & 4 & 0 \\ 5 & -2 & 3 \\ 2 & 1 & 4 \end{vmatrix}$ **c** $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 4 & 1 \\ 3 & 5 & 2 \end{vmatrix}$ **d** $\begin{vmatrix} 2 & -3 & 4 \\ 2 & 2 & 2 \\ 5 & 5 & 5 \end{vmatrix}$

7 Without using your calculator, find the values of these determinants.

a $\begin{vmatrix} 4 & 3 & -1 \\ 2 & -2 & 0 \\ 0 & 4 & -2 \end{vmatrix}$ **b** $\begin{vmatrix} 3 & -2 & 1 \\ 4 & 1 & -3 \\ 7 & 2 & -4 \end{vmatrix}$ **c** $\begin{vmatrix} 5 & -2 & -3 \\ 6 & 4 & 2 \\ -2 & -4 & -3 \end{vmatrix}$

8 The matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & -4 \\ 2k+1 & 3 & k \\ 1 & 0 & 1 \end{pmatrix}$.

Given that $\mathbf{A}$ is singular, find the value of the constant k.

9 The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ -2 & 1 & k+3 \end{pmatrix}$, where k is a constant.

Given that the determinant of $\mathbf{A}$ is 8, find the possible values of k.

10 The matrix $\mathbf{A} = \begin{pmatrix} 2 & 5 & 3 \\ -2 & 0 & 4 \\ 3 & 10 & 8 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & -2 & -1 \end{pmatrix}$.

a Show that $\mathbf{A}$ is singular.

b Find $\mathbf{AB}$.

c Show that $\mathbf{AB}$ is also singular.

11 Show that, for all values of a, b and c, the matrix $\begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}$ is singular. **(3 marks)**

12 Show that, for all real values of x, the matrix $\begin{pmatrix} 2 & -2 & 4 \\ 3 & x & -2 \\ -1 & 3 & x \end{pmatrix}$ is non-singular. **(3 marks)**

13 Find all the values of x for which the matrix $\begin{pmatrix} x-3 & -2 & 0 \\ 1 & x & -2 \\ -2 & -1 & x+1 \end{pmatrix}$ is singular. **(4 marks)**

14 The matrix $\mathbf{M} = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$ and the matrix $\mathbf{N} = \begin{pmatrix} -1 & k \\ 4 & 3 \end{pmatrix}$, where k is a constant.

a Evaluate the determinant of **M**. **(1 mark)**

b Given that the determinant of **N** is 7, find the value of k. **(2 marks)**

c Using the value of k found in part **b**, find **MN**. **(1 mark)**

d Verify that det **MN** = det **M** det **N**. **(1 mark)**

15 The matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 4 \\ -4 & 2 & 1 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 3 & 1 & 2 \\ k & 4 & 5 \\ 0 & 2 & 3 \end{pmatrix}$, where k is a constant.

a Evaluate the determinant of **A**. **(2 marks)**

Given that the determinant of **B** is 2,

b find the value of k. **(3 marks)**

Using the value of k found in part **b**,

c find **AB** **(2 marks)**

d verify that det **AB** = det **A** det **B**. **(2 marks)**

Exercise D

1 Determine which of these matrices are singular and which are non-singular. For those that are non-singular find the inverse matrix.

a $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & 5 \\ 0 & 0 \end{pmatrix}$

d $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ **e** $\begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$ **f** $\begin{pmatrix} 4 & 3 \\ 6 & 2 \end{pmatrix}$

2 Find inverses of these matrices, giving your answers in terms of a and b.

a $\begin{pmatrix} a & 1+a \\ 1+a & 2+a \end{pmatrix}$ **b** $\begin{pmatrix} 2a & 3b \\ -a & -b \end{pmatrix}$

3 **a** Given that **ABC** = **I**, prove that $\mathbf{B}^{-1} = \mathbf{CA}$.

b Given that $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & -6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}$, find **B**.

4 **a** Given that **AB** = **C**, find an expression for **B**, in terms of **A** and **C**.

b Given further that $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 & 6 \\ 1 & 22 \end{pmatrix}$, find **B**.

5 **a** Given that **BAC** = **B**, where **B** is a non-singular matrix, find an expression for **A** in terms of **C**.

b When $\mathbf{C} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$, find **A**.

6 The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$ and $\mathbf{AB} = \begin{pmatrix} 4 & 7 & -8 \\ -8 & -13 & 18 \end{pmatrix}$. Find the matrix **B**.

7 The matrix $\mathbf{B} = \begin{pmatrix} 5 & -4 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{AB} = \begin{pmatrix} 11 & -1 \\ -8 & 9 \\ -2 & -1 \end{pmatrix}$. Find the matrix **A**.

8 The matrix $\mathbf{A} = \begin{pmatrix} 3a & b \\ 4a & 2b \end{pmatrix}$, where a and b are non-zero constants.

a Find $\mathbf{A}^{-1}$, giving your answer in terms of a and b.

The matrix $\mathbf{B} = \begin{pmatrix} -a & b \\ 3a & 2b \end{pmatrix}$ and the matrix **X** is given by **B** = **XA**.

b Find **X**, giving your answer in terms of a and b.

9 The non-singular matrices **A** and **B** are such that **AB** = **BA**, and **ABA** = **B**.

a Prove that $\mathbf{A}^2 = \mathbf{I}$. **(3 marks)**

Given that $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, by considering a matrix **B** of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

b show that $a = d$ and $b = c$. **(3 marks)**

10 $\mathbf{M} = \begin{pmatrix} 2 & 3 \\ k & -1 \end{pmatrix}$ where k is a constant.

a For which values of k does **M** have an inverse? **(2 marks)**

b Given that **M** is non-singular, find $\mathbf{M}^{-1}$ in terms of k. **(3 marks)**

11 Given that $\mathbf{A} = \begin{pmatrix} 4 & p \\ -2 & -2 \end{pmatrix}$ where p is a constant and $p \neq 4$,

a find $\mathbf{A}^{-1}$ in terms of p. **(2 marks)**

b Given that $\mathbf{A} + \mathbf{A}^{-1} = \begin{pmatrix} 5 & \frac{9}{2} \\ -3 & -4 \end{pmatrix}$, find the value of p. **(3 marks)**

12 $\mathbf{M} = \begin{pmatrix} k & -3 \\ 4 & k+3 \end{pmatrix}$ where k is a real constant.

a Find det **M** in terms of k. **(2 marks)**

b Show that **M** is non-singular for all values of k. **(3 marks)**

c Given that $10\mathbf{M}^{-1} + \mathbf{M} = \mathbf{I}$ where **I** is the 2×2 identity matrix, find the value of k. **(3 marks)**

13 Given that $\mathbf{A} = \begin{pmatrix} a & 2 \\ 3 & 2a \end{pmatrix}$ where a is a real constant,

a find $\mathbf{A}^{-1}$ in terms of a **(3 marks)**

b write down two values of a for which $\mathbf{A}^{-1}$ does not exist. **(1 mark)**

Exercise E

1 Use your calculator to find the inverses of these matrices.

a $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

2 Without using a calculator, find the inverses of these matrices.

a $\begin{pmatrix} 1 & -3 & 2 \\ 0 & -2 & 1 \\ 3 & 0 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & 3 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ **c** $\begin{pmatrix} 3 & 2 & -7 \\ 1 & -3 & 1 \\ 0 & 2 & -2 \end{pmatrix}$

3 The matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

a Find $\mathbf{A}^{-1}$. **b** Find $\mathbf{B}^{-1}$.

Given that $(\mathbf{AB})^{-1} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

c verify that $\mathbf{B}^{-1}\mathbf{A}^{-1} = (\mathbf{AB})^{-1}$.

4 The matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & 3 \\ k & 1 & 1 \\ 1 & 1 & 4 \end{pmatrix}$.

a Show that $\det \mathbf{A} = 3(k + 1)$. **(3 marks)**

b Given that $k \neq -1$, find $\mathbf{A}^{-1}$. **(4 marks)**

5 The matrix $\mathbf{A} = \begin{pmatrix} 5 & a & 4 \\ b & -7 & 8 \\ 2 & -2 & c \end{pmatrix}$.

Given that $\mathbf{A} = \mathbf{A}^{-1}$, find the values of the constants a, b and c. **(6 marks)**

6 The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}$.

a Show that $\mathbf{A}^3 = \mathbf{I}$. **b** Hence find $\mathbf{A}^{-1}$.

7 The matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & -3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$.

a Show that $\mathbf{A}^3 = 13\mathbf{A} - 15\mathbf{I}$. **b** Deduce that $15\mathbf{A}^{-1} = 13\mathbf{I} - \mathbf{A}^2$. **c** Hence find $\mathbf{A}^{-1}$.

8 The matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 3 & -2 \\ 0 & 3 & -4 \end{pmatrix}$.

a Show that $\mathbf{A}$ is singular.

The matrix $\mathbf{C}$ is the matrix of the cofactors of $\mathbf{A}$.

b Find $\mathbf{C}$. **c** Show that $\mathbf{AC}^{\mathrm{T}} = \mathbf{0}$.

9 $\mathbf{M} = \begin{pmatrix} 2 & k & 3 \\ -1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}$ where k is a real constant.

a For which values of k does $\mathbf{M}$ have an inverse? **(2 marks)**

b Given that $\mathbf{M}$ is non-singular, find $\mathbf{M}^{-1}$ in terms of k. **(4 marks)**

10 $\mathbf{A} = \begin{pmatrix} p & 2p & 3 \\ 4 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$ where p is a real constant.

Given that $\mathbf{A}$ is non-singular, find $\mathbf{A}^{-1}$ in terms of p. **(4 marks)**

Exercise F

1 Solve the following systems of equations using inverse matrices.

a $\begin{aligned} 2x - 6y + 4z &= 32 \\ 3x + 2y - 9z &= -49 \\ -2x + 4y + z &= -3 \end{aligned}$ **b** $\begin{aligned} -4x + 6y - 2z &= -22 \\ 3x + 3y - 2z &= 1 \\ -6x - 7y + 3z &= 3 \end{aligned}$

c $\begin{aligned} 4x + 7y - 2z &= 21 \\ -10x - y - 7z &= 0 \\ -2x + y - 4z &= 9 \end{aligned}$ **d** $\begin{aligned} -3x - 6y + 4z &= -23 \\ -3x + 6y - 10z &= -1 \\ 3x + 7y - 3z &= 27 \end{aligned}$

2 Three planes A, B and C are defined by the following equations.

A: $x - 3y - 4z = 3$
B: $6x + 5y - 7z = 30$
C: $x + 4y + 6z = -3$

By constructing and solving a suitable matrix equation, show that these three planes intersect at a single point and find the coordinates of that point. **(5 marks)**

3 Phyllis invested £3000 across three savings accounts, A, B and C. She invested £190 more in account A than in account B.
After two years, account A had increased in value by 1%, account B had increased in value by 2.5% and account C had decreased in value by 1.5%. The total value of Phyllis's savings had increased by £41.
Form and solve a matrix equation to find out how much money was invested by Phyllis in each account. **(7 marks)**

4 A colony of bats is made up of brown bats, grey bats and black bats. Initially there are 2000 bats and there are 250 more brown bats than grey bats.
After one year:

- the number of brown bats had fallen by 1%
- the number of grey bats had fallen by 2%
- the number of black bats had increased by 4%
- overall there were 40 more bats

Form and solve a matrix equation to find out how many of each colour bat there were in the initial colony. **(7 marks)**

5 Three planes A, B and C are defined by the following equations:

$$A: \quad x + ay + 2z = a$$
$$B: \quad x - y - z = a$$
$$C: \quad x + 4y + 4z = 0$$

Given that the planes do not meet at a single point,

a find the value of a **(4 marks)**

b determine whether the three equations form a consistent system, and give a geometric interpretation of your answer. **(4 marks)**

Hint If the three planes do not meet at a single point, the corresponding 3×3 matrix must be singular.

6 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 1 & 4 & q \\ 2 & 3 & -3 \\ q & q & -2 \end{pmatrix}$

a Given that $\det \mathbf{M} = 0$, show that $q^2 + 9q - 10 = 0$. **(4 marks)**

A system of simultaneous equations is shown below:

$$x + 4y + qz = -16$$
$$2x + 3y - 3z = \tfrac{1}{2}q$$
$$qx + qy - 2z = -2$$

b For each of the following values of q, determine whether the system of equations is consistent or inconsistent. If the system is consistent, determine whether there is a unique solution or an infinity of solutions. In each case, identify the geometric configuration of the planes corresponding to each equation.

i $q = -10$ **ii** $q = 2$ **iii** $q = 1$ **(7 marks)**

Exercise G

1 The matrix $\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{AB} = \begin{pmatrix} 4 & 1 & 9 \\ 1 & 9 & 4 \end{pmatrix}$. Find the matrix $\mathbf{B}$.

2 The matrix $\mathbf{A} = \begin{pmatrix} a & b \\ 2a & 3b \end{pmatrix}$, where a and b are non-zero constants.

a Find $\mathbf{A}^{-1}$, giving your answer in terms of a and b. **(2 marks)**

The matrix $\mathbf{Y} = \begin{pmatrix} a & 2b \\ 2a & b \end{pmatrix}$ and the matrix $\mathbf{X}$ is given by $\mathbf{XA} = \mathbf{Y}$.

b Find $\mathbf{X}$, giving your answer in terms of a and b. **(3 marks)**

3 The 2×2, non-singular matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{X}$ satisfy $\mathbf{XB} = \mathbf{BA}$.

a Find an expression for $\mathbf{X}$ in terms of $\mathbf{A}$ and $\mathbf{B}$. **(1 mark)**

b Given that $\mathbf{A} = \begin{pmatrix} 5 & 3 \\ 0 & -2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$, find $\mathbf{X}$. **(2 marks)**

4 A matrix $\mathbf{A}$ is given as $\mathbf{A} = \begin{pmatrix} a & 2 & -1 \\ -1 & 1 & -1 \\ b & 2 & 1 \end{pmatrix}$.

Given that $\mathbf{A}^2 = \begin{pmatrix} -4 & 2 & -4 \\ -5 & -3 & -1 \\ 4 & 10 & -4 \end{pmatrix}$, find the values of a and b. **(3 marks)**

5 $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ t & 3 & 1 \\ -2 & -1 & 1 \end{pmatrix}$

Given that $\mathbf{A}$ is singular, find the value of t. **(3 marks)**

6 $\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ x & 2 & 0 \\ 3 & 1 & 1 \end{pmatrix}$

Find $\mathbf{M}^{-1}$ in terms of x. **(4 marks)**

7 $\mathbf{A} = \begin{pmatrix} k & -2 \\ -4 & k \end{pmatrix}$ where k is a real constant.

a For which values of k does $\mathbf{A}$ have an inverse? **(2 marks)**

b Given that $\mathbf{A}$ is non-singular, find $\mathbf{A}^{-1}$ in terms of k. **(3 marks)**

8 $\mathbf{B} = \begin{pmatrix} k & 6 \\ -1 & k-2 \end{pmatrix}$ where k is a real constant.

a Find det $\mathbf{B}$ in terms of k. **(2 marks)**

b Show that $\mathbf{B}$ is non-singular for all values of k. **(3 marks)**

c Given that $21\mathbf{B}^{-1} + \mathbf{B} = -8\mathbf{I}$ where $\mathbf{I}$ is the 2×2 identity matrix, find the value of k. **(3 marks)**

9 Given that $\mathbf{M} = \begin{pmatrix} 2 & -m \\ m & -1 \end{pmatrix}$ where m is a real constant,

a write down two values of m such that $\mathbf{M}$ is singular **(2 marks)**

b find $\mathbf{M}^{-1}$ in terms of m, given that $\mathbf{M}$ is non-singular. **(3 marks)**

10 $\mathbf{A} = \begin{pmatrix} 3 & 4 & 5 \\ 1 & a & -1 \\ -2 & 1 & 1 \end{pmatrix}$ where a is a real constant.

a For which values of a does $\mathbf{A}$ have an inverse? **(2 marks)**

b Given that $\mathbf{A}$ is non-singular, find $\mathbf{A}^{-1}$ in terms of a. **(4 marks)**

11 Three planes A, B and C are defined by the following equations.

A: $x + y + z = 6$

B: $x - 4y + 2z = -2$

C: $2x + y - 3z = 0$

By constructing and solving a suitable matrix equation, show that these three planes intersect at a single point and find the coordinates of that point. **(5 marks)**

12 A sheep farmer has three types of sheep: Hampshire, Dorset horn and Wiltshire horn. Initially his flock had 2500 sheep in it. There were 300 more Hampshire sheep than Wiltshire horn.

After one year:

- the number of Hampshire sheep had increased by 6%
- the number of Dorset horn had increased by 4%
- the number of Wiltshire horn had increased by 3%
- overall the flock size had increased by 110

Form and solve a matrix equation to find out how many of each type of sheep there were in the initial flock. **(7 marks)**

13 a Determine the values of the real constants a and b for which there are infinitely many solutions to the simultaneous equations

$$2x + 3y + z = 6$$
$$-x + y + 2z = 7$$
$$ax + y + 4z = b$$

(6 marks)

b Give a geometric interpretation of the three planes formed by these equations.

(1 mark)