Exponentials and Logarithms 2

Exercise A

1 Sketch the graphs of

a $y = e^x + 1$ **b** $y = 4e^{-2x}$ **c** $y = 2e^x - 3$

d $y = 4 - e^x$ **e** $y = 6 + 10e^{\frac{1}{2}x}$ **f** $y = 100e^{-x} + 10$

2 The value of a car varies according to the formula

$$V = 20\,000\,e^{-\frac{t}{12}}$$

where V is the value in £'s and t is its age in years from new.

a State its value when new.

b Find its value (to the nearest £) after 4 years.

c Sketch the graph of V against t.

3 The population of a country is increasing according to the formula

$$P = 20 + 10\,e^{\frac{t}{50}}$$

where P is the population in thousands and t is the time in years after the year 2000.

a State the population in the year 2000.

b Use the model to predict the population in the year 2020.

c Sketch the graph of P against t for the years 2000 to 2100.

4 The number of people infected with a disease varies according to the formula

$$N = 300 - 100\,e^{-0.5t}$$

where N is the number of people infected with the disease and t is the time in years after detection.

a How many people were first diagnosed with the disease?

b What is the long term prediction of how this disease will spread?

c Graph N against t.

5 The value of an investment varies according to the formula

$$V = A\,e^{\frac{t}{12}}$$

where V is the value of the investment in £'s, A is a constant to be found and t is the time in years after the investment was made.

a If the investment was worth £8000 after 3 years find A to the nearest £.

b Find the value of the investment after 10 years.

c By what factor will be the original investment have increased by after 20 years?

(More Questions on Page 2)

Exercise B

1 Solve the following equations giving exact solutions:

a $e^x = 5$ **b** $\ln x = 4$ **c** $e^{2x} = 7$

d $\ln \frac{x}{2} = 4$ **e** $e^{x-1} = 8$ **f** $\ln(2x + 1) = 5$

g $e^{-x} = 10$ **h** $\ln(2 - x) = 4$ **i** $2e^{4x} - 3 = 8$

2 Solve the following giving your solution in terms of ln 2:

a $e^{3x} = 8$ **b** $e^{-2x} = 4$ **c** $e^{2x+1} = 0.5$

3 Sketch the following graphs stating any asymptotes and intersections with axes:

a $y = \ln(x + 1)$ **b** $y = 2\ln x$ **c** $y = \ln(2x)$

d $y = (\ln x)^2$ **e** $y = \ln(4 - x)$ **f** $y = 3 + \ln(x + 2)$

4 The price of a new car varies according to the formula

$$P = 15\,000\,e^{-\frac{t}{10}}$$

where P is the price in £'s and t is the age in years from new.

a State its new value.

b Calculate its value after 5 years (to the nearest £).

c Find its age when its price falls below £5000.

d Sketch the graph showing how the price varies over time. Is this a good model?

5.

The number of bacteria in a culture grows according to the following equation:

$$N = 100 + 50\,e^{\frac{t}{30}}$$

where N is the number of bacteria present and t is the time in days from the start of the experiment.

a State the number of bacteria present at the start of the experiment.

b State the number after 10 days.

c State the day on which the number first reaches 1 000 000.

d Sketch the graph showing how N varies with t.

Exercise C

1 Sketch the following functions stating any asymptotes and intersections with axes:

a $y = e^x + 3$ **b** $y = \ln(-x)$ **c** $y = \ln(x + 2)$

d $y = 3e^{-2x} + 4$ **e** $y = e^{x+2}$ **f** $y = 4 - \ln x$

2 Solve the following equations, giving exact solutions:

a $\ln(2x - 5) = 8$ **b** $e^{4x} = 5$ **c** $24 - e^{-2x} = 10$

d $\ln x + \ln(x - 3) = 0$ **e** $e^x + e^{-x} = 2$ **f** $\ln 2 + \ln x = 4$

3.

The price of a computer system can be modelled by the formula

$$P = 100 + 850\,e^{-\frac{t}{2}}$$

where P is the price of the system in £s and t is the age of the computer in years after being purchased.

a Calculate the new price of the system.

b Calculate its price after 3 years.

c When will it be worth less than £200?

d Find its price as $t \to \infty$.

e Sketch the graph showing P against t.

Comment on the appropriateness of this model.

Exercise D

1.

A particular species of orchid is being studied. The population p at time t years after the study started is assumed to be

$$p = \frac{2800ae^{0.2t}}{1 + ae^{0.2t}}, \text{ where } a \text{ is a constant.}$$

Given that there were 300 orchids when the study started,

(*a*) show that $a = 0.12$,

(3)

(*b*) use the equation with $a = 0.12$ to predict the number of years before the population of orchids reaches 1850.

(4)

(*c*) Show that $p = \dfrac{336}{0.12 + e^{-0.2t}}$.

(1)

(*d*) Hence show that the population cannot exceed 2800.

(2)

2.

A heated metal ball is dropped into a liquid. As the ball cools, its temperature, T °C, t minutes after it enters the liquid, is given by

$$T = 400e^{-0.05t} + 25, \quad t \geq 0.$$

(*a*) Find the temperature of the ball as it enters the liquid.

(1)

(*b*) Find the value of t for which $T = 300$, giving your answer to 3 significant figures.

(4)

(*c*) Find the rate at which the temperature of the ball is decreasing at the instant when $t = 50$. Give your answer in °C per minute to 3 significant figures.

(3)

(*d*) From the equation for temperature T in terms of t, given above, explain why the temperature of the ball can never fall to 20 °C.

(1)

3.

The amount of a certain type of drug in the bloodstream t hours after it has been taken is given by the formula

$$x = De^{-\frac{1}{8}t},$$

where x is the amount of the drug in the bloodstream in milligrams and D is the dose given in milligrams.

A dose of 10 mg of the drug is given.

(a) Find the amount of the drug in the bloodstream 5 hours after the dose is given. Give your answer in mg to 3 decimal places.

(2)

A second dose of 10 mg is given after 5 hours.

(b) Show that the amount of the drug in the bloodstream 1 hour after the second dose is 13.549 mg to 3 decimal places.

(2)

No more doses of the drug are given. At time T hours after the second dose is given, the amount of the drug in the bloodstream is 3 mg.

(c) Find the value of T.

(3)

4.

The radioactive decay of a substance is given by

$$R = 1000e^{-ct}, \qquad t \geqslant 0.$$

where R is the number of atoms at time t years and c is a positive constant.

(a) Find the number of atoms when the substance started to decay.

(1)

It takes 5730 years for half of the substance to decay.

(b) Find the value of c to 3 significant figures.

(4)

(c) Calculate the number of atoms that will be left when $t = 22\ 920$.

(2)

(d) In the space provided on page 13, sketch the graph of R against t .

(2)

5.

Find the exact solutions to the equations

(a) $\ln x + \ln 3 = \ln 6$,

(2)

(b) $e^x + 3e^{-x} = 4$.

(4)

6.

A heated metal ball is dropped into a liquid. As the ball cools, its temperature, T °C, t minutes after it enters the liquid, is given by

$$T = 400e^{-0.05t} + 25, \quad t \geq 0.$$

(*a*) Find the temperature of the ball as it enters the liquid.

(1)

(*b*) Find the value of t for which $T = 300$, giving your answer to 3 significant figures.

(4)

(*c*) Find the rate at which the temperature of the ball is decreasing at the instant when $t = 50$. Give your answer in °C per minute to 3 significant figures.

(3)

(*d*) From the equation for temperature T in terms of t, given above, explain why the temperature of the ball can never fall to 20 °C.

(1)

7.

Rabbits were introduced onto an island. The number of rabbits, P, t years after they were introduced is modelled by the equation

$$P = 80e^{\frac{1}{5}t}, \qquad t \in \mathbb{R},\ t \geqslant 0$$

(a) Write down the number of rabbits that were introduced to the island.

(1)

(b) Find the number of years it would take for the number of rabbits to first exceed 1000.

(2)

(c) Find $\dfrac{dP}{dt}$.

(2)

(d) Find P when $\dfrac{dP}{dt} = 50$.

(3)

8.

Joan brings a cup of hot tea into a room and places the cup on a table. At time t minutes after Joan places the cup on the table, the temperature, θ°C, of the tea is modelled by the equation

$$\theta = 20 + Ae^{-kt},$$

where A and k are positive constants.

Given that the initial temperature of the tea was 90°C,

(a) find the value of A.

(2)

The tea takes 5 minutes to decrease in temperature from 90°C to 55°C.

(b) Show that $k = \dfrac{1}{5}\ln 2$.

(3)

(c) Find the rate at which the temperature of the tea is decreasing at the instant when $t = 10$. Give your answer, in °C per minute, to 3 decimal places.

(3)

9.

The mass, m grams, of a leaf t days after it has been picked from a tree is given by

$$m = p\,e^{-kt}$$

where k and p are positive constants.

When the leaf is picked from the tree, its mass is 7.5 grams and 4 days later its mass is 2.5 grams.

(a) Write down the value of p.

(1)

(b) Show that $k = \frac{1}{4}\ln 3$.

(4)

(c) Find the value of t when $\frac{dm}{dt} = -0.6\ln 3$.

(6)

10.

The value of Bob's car can be calculated from the formula

$$V = 17000e^{-0.25t} + 2000e^{-0.5t} + 500$$

where V is the value of the car in pounds (£) and t is the age in years.

(a) Find the value of the car when $t = 0$

(1)

(b) Calculate the exact value of t when $V = 9500$

(4)

(c) Find the rate at which the value of the car is decreasing at the instant when $t = 8$. Give your answer in pounds per year to the nearest pound.

(4)

11.

The mass, m grams, of a substance at time t years is given by the formula

$$m = 180e^{-0.017t}.$$

(i) Find the value of t for which the mass is 25 grams. [3]

(ii) Find the rate at which the mass is decreasing when $t = 55$. [3]

12.

(a)

t	0	10	20
X	275	440	

The quantity X is increasing exponentially with respect to time t. The table above shows values of X for different values of t. Find the value of X when $t = 20$. [3]

(b) The quantity Y is decreasing exponentially with respect to time t where

$$Y = 80e^{-0.02t}.$$

(i) Find the value of t for which $Y = 20$, giving your answer correct to 2 significant figures. [3]

(ii) Find by differentiation the rate at which Y is decreasing when $t = 30$, giving your answer correct to 2 significant figures. [3]

13.

A substance is decaying in such a way that its mass, m kg, at a time t years from now is given by the formula

$$m = 240e^{-0.04t}.$$

(i) Find the time taken for the substance to halve its mass. [3]

(ii) Find the value of t for which the mass is decreasing at a rate of 2.1 kg per year. [4]

14.

The mass, M grams, of a certain substance is increasing exponentially so that, at time t hours, the mass is given by

$$M = 40e^{kt},$$

where k is a constant. The following table shows certain values of t and M.

t	0	21	63
M		80	

(i) In either order,

(a) find the values missing from the table, **[3]**

(b) determine the value of k. **[2]**

(ii) Find the rate at which the mass is increasing when $t = 21$. **[3]**

15.

An experiment involves two substances, Substance 1 and Substance 2, whose masses are changing. The mass, M_1 grams, of Substance 1 at time t hours is given by

$$M_1 = 400e^{-0.014t}.$$

The mass, M_2 grams, of Substance 2 is increasing exponentially and the mass at certain times is shown in the following table.

t (hours)	0	10	20
M_2 (grams)	75	120	192

A critical stage in the experiment is reached at time T hours when the masses of the two substances are equal.

(i) Find the rate at which the mass of Substance 1 is decreasing when $t = 10$, giving your answer in grams per hour correct to 2 significant figures. **[3]**

(ii) Show that T is the root of an equation of the form $e^{kt} = c$, where the values of the constants k and c are to be stated. **[5]**

(iii) Hence find the value of T correct to 3 significant figures. **[2]**