Numerical Methods

Exercise A

1 Show that each of these functions has at least one root in the given interval.

a $f(x) = x^3 - x + 5, -2 < x < -1$

b $f(x) = x^2 - \sqrt{x} - 10, 3 < x < 4$

c $f(x) = x^3 - \frac{1}{x} - 2, -0.5 < x < -0.2$

d $f(x) = e^x - \ln x - 5, 1.65 < x < 1.75$

2 $f(x) = 3 + x^2 - x^3$

a Show that the equation $f(x) = 0$ has a root, α, in the interval [1.8, 1.9]. **(2 marks)**

b By considering a change of sign of $f(x)$ in a suitable interval, verify that $\alpha = 1.864$ correct to 3 decimal places. **(3 marks)**

3 $h(x) = \sqrt[3]{x} - \cos x - 1$, where x is in radians.

a Show that the equation $h(x) = 0$ has a root, α, between $x = 1.4$ and $x = 1.5$. **(2 marks)**

b By choosing a suitable interval, show that $\alpha = 1.441$ is correct to 3 decimal places. **(3 marks)**

Exercise B

1 $f(x) = x^2 - 6x + 2$

a Show that $f(x) = 0$ can be written as:

i $x = \frac{x^2 + 2}{6}$ **ii** $x = \sqrt{6x - 2}$ **iii** $x = 6 - \frac{2}{x}$

b Starting with $x_0 = 4$, use each iterative formula to find a root of the equation $f(x) = 0$. Round your answers to 3 decimal places.

c Use the quadratic formula to find the roots to the equation $f(x) = 0$, leaving your answer in the form $a \pm \sqrt{b}$, where a and b are constants to be found.

2 $f(x) = x^2 - 5x - 3$

a Show that $f(x) = 0$ can be written as:

i $x = \sqrt{5x + 3}$ **ii** $x = \frac{x^2 - 3}{5}$

b Let $x_0 = 5$. Show that each of the following iterative formulae gives different roots of $f(x) = 0$.

i $x_{n+1} = \sqrt{5x_n + 3}$ **ii** $x_{n+1} = \frac{x_n^2 - 3}{5}$

3 $f(x) = x^2 - 6x + 1$

a Show that the equation $f(x) = 0$ can be written as $x = \sqrt{6x - 1}$. **(1 mark)**

b Sketch on the same axes the graphs of $y = x$ and $y = \sqrt{6x - 1}$. **(2 marks)**

c Write down the number of roots of $f(x)$. **(1 mark)**

d Use your diagram to explain why the iterative formula $x_{n+1} = \sqrt{6x_n - 1}$ converges to a root of $f(x)$ when $x_0 = 2$. **(1 mark)**

$f(x) = 0$ can also be rearranged to form the iterative formula $x_{n+1} = \frac{x_n^2 + 1}{6}$.

e By sketching a diagram, explain why the iteration diverges when $x_0 = 10$. **(2 marks)**

Exercise C

1 $f(x) = x^3 - 2x - 1$

a Show that the equation $f(x) = 0$ has a root, α, in the interval $1 < \alpha < 2$.

b Using $x_0 = 1.5$ as a first approximation to α, apply the Newton–Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places.

2 $f(x) = x^2 - \frac{4}{x} + 6x - 10, x \neq 0.$

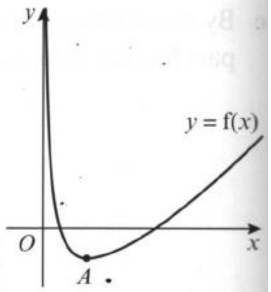
a Use differentiation to find $f'(x)$. **(2 marks)**

The root, α, of the equation $f(x) = 0$ lies in the interval $[-0.4, -0.3]$.

b Taking -0.4 as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places. **(4 marks)**

3 The diagram shows part of the curve with equation

$y = f(x)$, where $f(x) = x^{\frac{3}{2}} - e^{-x} + \frac{1}{\sqrt{x}} - 2, x > 0.$



The point A, with x-coordinate q, is a stationary point on the curve.

The equation $f(x) = 0$ has a root α in the interval $[1.2, 1.3]$.

a Explain why $x_0 = q$ is not suitable to use as a first approximation when applying the Newton–Raphson method. **(1 mark)**

b Taking $x_0 = 1.2$ as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places. **(4 marks)**

4 $f(x) = 1 - x - \cos(x^2)$

a Show that the equation $f(x) = 0$ has a root α in the interval $1.4 < \alpha < 1.5$. **(1 mark)**

b Using $x_0 = 1.4$ as a first approximation to α, apply the Newton–Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places. **(4 marks)**

c By considering the change of sign of $f(x)$ over an appropriate interval, show that your answer to part **b** is correct to 3 decimal places. **(2 marks)**

5 $f(x) = x^2 - \frac{3}{x^2}, x \geqslant 0$

a Show that a root α of the equation $f(x) = 0$ lies in the interval $[1.3, 1.4]$. **(1 mark)**

b Differentiate $f(x)$ to find $f'(x)$. **(2 marks)**

c By taking 1.3 as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places. **(3 marks)**