18.4 Kepler's laws

Specification reference: 5.4.3

Back into darkness

Comet Lovejoy (Figure 1) is one of the most recently discovered great comets. Great comets appear only once a decade or so, and are so bright they are clearly visible at night, and sometimes even during the day.

The shape of the orbit of the comet, like that of all bodies orbiting the Sun, is governed by Kepler's laws of planetary motion. The laws explain why the comet travels much faster when it is closer to the Sun, giving us only a few months to enjoy its brilliance before it moves away, spending centuries in orbit beyond even the most distant planets.

Kepler's laws of planetary motion

Johannes Kepler was a brilliant German astronomer and mathematician. In the early years of the 17th century he published his three laws of planetary motion. These laws were based purely on observational data for the then known planets. Kepler had no knowledge of gravitational fields – his ideas helped Newton to formulate his law of gravitation.

Kepler's first law: The orbit of a planet is an **ellipse** with the Sun at one of the two foci (Figure 2).

An ellipse is a 'squashed' or elongated circle, with two foci. The orbits of all the planets are elliptical.

In most cases the orbits have a low **eccentricity** (a measure of how elongated the circle is), and so their orbits are modelled as circles. For example, at **aphelion** (the furthest point from the Sun) the Earth is 152 million km from the Sun, but at **perihelion** (the closest point to the Sun) the distance is 147 million km, a change of just 3%.

Kepler's second law: A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.

As planets move on their elliptical orbit around the Sun, their speed is not constant. When a planet is closer to the Sun it moves faster. In Figure 3, between X and Y the planet moves faster than between P and Q. Kepler's second law states if the time interval from X to Y is the same as for P to Q (e.g., 1 month) the areas A and B must be the same.

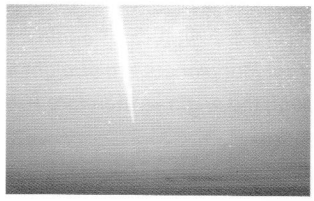
This helps explain why we rarely see great comets. Their orbits are highly elliptical, and when they get close to the Sun, where we can see them, they move fast and so spend much less time on this part of their orbit than far away from the Sun. Comets spend most of their time too far from the Sun to be visible.

Kepler's third law: The square of the orbital period T of a planet is directly proportional to the cube of its average distance r from the Sun.

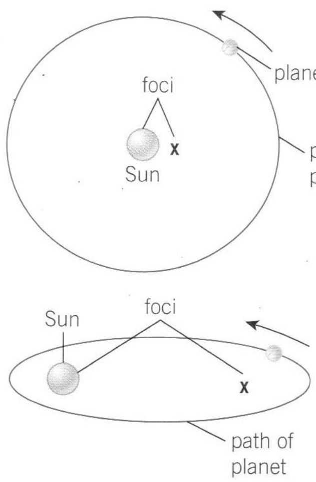
Learning outcomes

Demonstrate knowledge, understanding, and application of:

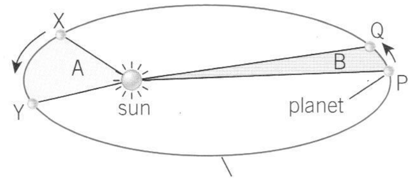
- → Kepler's three laws of planetary motion
- → the centripetal force on a planet from the gravitational force between it and the Sun
- → the equation $T^2 = \left(\frac{4\pi^2}{GM}\right)r^3$
- → the relationship for Kepler's third law $T^2 \propto r^3$ applied to systems other than our Solar System.



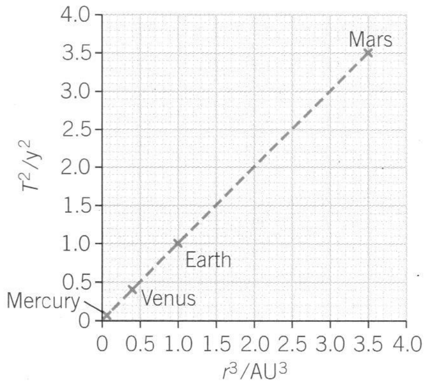
▲ **Figure 1** *Comet Lovejoy was a spectacular sight in 2011 and will next be visible in 2633 as it has an orbital period of 622 years*



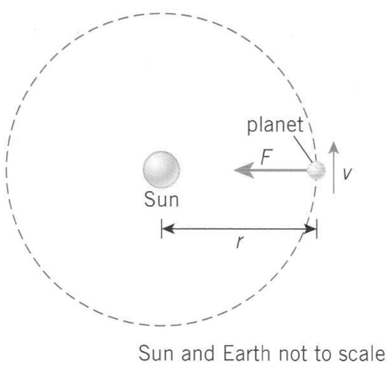
▲ **Figure 2** *The orbits of all planets are elliptical, but most of these ellipses are close to circles – here the left orbit is nearly circular whilst the right-hand orbit is highly elliptical*



▲ **Figure 3** *Kepler's second law describes the motion of a planet in its elliptical path around the Sun*



▲ **Figure 4** *A graph of T^2 against r^3 for the first four planets is a straight line through the origin — therefore $T^2 \propto r^3$*



▲ **Figure 5** *Circular motion and Newton's law of gravitation are used in modelling orbits as circles*

Synoptic link

The mathematics of circular motion was studied in Chapter 16.

This can be written as a relationship as $T^2 \propto r^3$ or as

$$\frac{T^3}{r^3} = k$$

where k is a constant for the planets orbiting the Sun.

Table 1 shows data for the six inner planets in our Solar System. The values for T and r are relative to the values of T and r for Earth, 1.00 year and 1.50×10^{11} m, respectively. The mean distance between the Earth and the Sun is known as one **astronomical unit** (AU).

▼ **Table 1** *Data for the six inner planets*

Planet	T/y	T^2/y^2	r/AU	r^3/AU^3	$\frac{T^2}{r^3}$/y^2 AU^{-3}
Mercury	0.24	0.058	0.40	0.064	0.91
Venus	0.62	0.38	0.70	0.343	1.11
Earth	1.00	1.00	1.00	1.00	1.00
Mars	1.88	3.53	1.50	3.37	1.05
Jupiter	11.86	140.6	5.20	141	1.00
Saturn	29.46	867.9	9.50	857	1.01

In the units y^2 AU^{-3}, the ratio $\frac{T^2}{r^3}$ for the planets in the Solar System is approximately 1 — Kepler's third law is validated.

Modelling planetary orbits as circles

Most planets in the Solar System have almost circular orbits. We can therefore use the mathematics developed for circular motion along with Newton's law of gravitation to relate the orbital period T of a planet to its distance r from the Sun. In doing so, we can provide theoretical justification for Kepler's empirical third law.

Consider a planet of mass m orbiting the Sun at a distance r. The orbital speed of the planet is v and it has an orbital period T. The mass of the Sun is M. The centripetal force on the planet is provided by the gravitational force between it and the Sun. Therefore, the gravitational force F on the planet must be equal to the centripetal force. That is

centripetal force on planet = gravitational force on planet

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$\text{or} \quad v^2 = \frac{GM}{r}$$

Since the planet is moving in a circle, the speed v of the planet can be determined by dividing the circumference of its orbit by its orbital period, $v = \frac{2\pi r}{T}$. Substituting this into the equation above gives

$$\frac{4\pi^2 r^2}{T^2} = \frac{GM}{r}$$

This can be rearranged to give

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$