Differential Equations

1.

Find the particular solution of each of the following differential equations.

(i) $\dfrac{dy}{dx} = x^2 - 1$ $\quad y = 2$ when $x = 3$

(ii) $\dfrac{dy}{dx} = x^2y$ $\quad y = 1$ when $x = 0$

(iii) $\dfrac{dy}{dx} = xe^{-y}$ $\quad y = 0$ when $x = 0$

(iv) $\dfrac{dy}{dx} = y^2$ $\quad y = 1$ when $x = 1$

(v) $\dfrac{dy}{dx} = x(y + 1)$ $\quad y = 0$ when $x = 1$

(vi) $\dfrac{dy}{dx} = y^2 \sin x$ $\quad y = 1$ when $x = 0$

2.

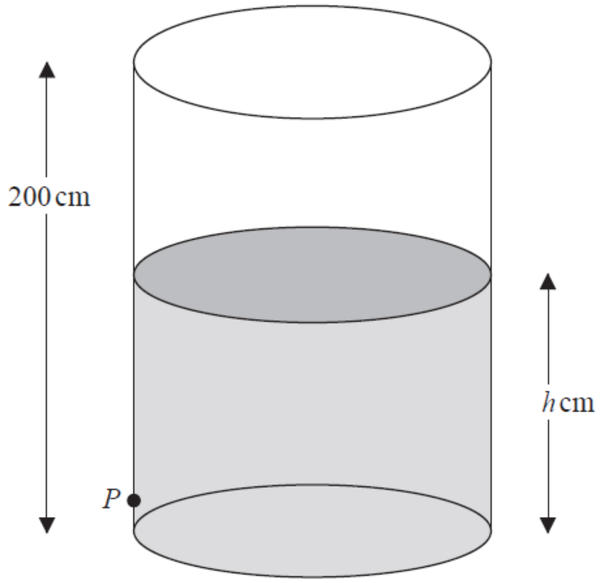

Diagram not drawn to scale

Figure 3

Figure 3 shows a vertical cylindrical tank of height 200 cm containing water. Water is leaking from a hole P on the side of the tank.

At time t minutes after the leaking starts, the height of water in the tank is h cm.

The height h cm of the water in the tank satisfies the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = k(h-9)^{\frac{1}{2}}, \qquad 9 < h \leqslant 200$$

where k is a constant.

Given that, when $h = 130$, the height of the water is falling at a rate of 1.1 cm per minute,

(a) find the value of k.

(2)

Given that the tank was full of water when the leaking started,

(b) solve the differential equation with your value of k, to find the value of t when $h = 50$

(6)

3.

Solve the differential equation

$$e^{2y}\frac{\mathrm{d}y}{\mathrm{d}x} + \tan x = 0,$$

given that $x = 0$ when $y = 0$. Give your answer in the form $y = \mathrm{f}(x)$. **[6]**

4.

The temperature of a freezer is –20 °C. A container of a liquid is placed in the freezer. The rate at which the temperature, θ °C, of a liquid decreases is proportional to the difference in temperature between the liquid and its surroundings. The situation is modelled by the differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta + 20),$$

where time t is in minutes and k is a positive constant.

(i) Express θ in terms of t, k and an arbitrary constant. **[3]**

Initially the temperature of the liquid in the container is 40 °C and, at this instant, the liquid is cooling at a rate of 3 °C per minute. The liquid freezes at 0 °C.

(ii) Find the value of k and find also the time it takes (to the nearest minute) for the liquid to freeze. **[5]**

The procedure is repeated on another occasion with a different liquid. The initial temperature of this liquid is 90 °C. After 19 minutes its temperature is 0 °C.

(iii) Without any further calculation, explain what you can deduce about the value of k in this case. **[1]**

5.

(a) Express $\dfrac{2}{P(P-2)}$ in partial fractions.

(3)

A team of biologists is studying a population of a particular species of animal.

The population is modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{2}P(P-2)\cos 2t, \quad t \geqslant 0$$

where P is the population in thousands, and t is the time measured in years since the start of the study.

Given that $P = 3$ when $t = 0$,

(b) solve this differential equation to show that

$$P = \frac{6}{3 - e^{\frac{1}{2}\sin 2t}}$$

(7)

(c) find the time taken for the population to reach 4000 for the first time.
Give your answer in years to 3 significant figures.

(3)

6.

The total value of the sales made by a new company in the first t years of its existence is denoted by £V. A model is proposed in which the rate of increase of V is proportional to the square root of V. The constant of proportionality is k.

(i) Express the model as a differential equation.

Verify by differentiation that $V = (\frac{1}{2}kt + c)^2$, where c is an arbitrary constant, satisfies this differential equation. **[4]**

(ii) The value of the company's sales in its first year is £10 000, and the total value of the sales in the first two years is £40 000. Find V in terms of t. **[4]**

7.

At time t seconds, the radius of a spherical balloon is r cm. The balloon is being inflated so that the rate of increase of its radius is inversely proportional to the square root of its radius. When $t = 5$, $r = 9$ and, at this instant, the radius is increasing at 1.08 cm s^{-1}.

(i) Write down a differential equation to model this situation, and solve it to express r in terms of t. **[7]**

(ii) How much air is in the balloon initially? **[2]**

[The volume of a sphere is $V = \frac{4}{3}\pi r^3$.]

8.

A container in the shape of an inverted cone of radius 3 metres and vertical height 4.5 metres is initially filled with liquid fertiliser. This fertiliser is released through a hole in the bottom of the container at a rate of 0.01 m^3 per second. At time t seconds the fertiliser remaining in the container forms an inverted cone of height h metres.

[The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.]

(i) Show that $h^2 \frac{dh}{dt} = -\frac{9}{400\pi}$. **[5]**

(ii) Express h in terms of t. **[4]**

(iii) Find the time it takes to empty the container, giving your answer to the nearest minute. **[2]**

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9. The temperature of a hot liquid decreases at a rate proportional to the temperature difference between the liquid and the surroundings. The Temperature of the surroundings is constant at 20^0C.

At the beginning the temperature of the liquid is 80^0C and after 5 minutes it is 70^0C.

Set up a differential equation and solve it to find an equation for the temperature in terms of time.
