Trigonometry (Year 13)

Exercise A

1 Rewrite the following as powers of $\sec\theta$, $\operatorname{cosec}\theta$ or $\cot\theta$:

a $\dfrac{1}{\sin^3\theta}$ **b** $\sqrt{\dfrac{4}{\tan^6\theta}}$ **c** $\dfrac{1}{2\cos^2\theta}$ **d** $\dfrac{1-\sin^2\theta}{\sin^2\theta}$

e $\dfrac{\sec\theta}{\cos^4\theta}$ **f** $\sqrt{\operatorname{cosec}^3\theta\cot\theta\sec\theta}$ **g** $\dfrac{2}{\sqrt{\tan\theta}}$ **h** $\dfrac{\operatorname{cosec}^2\theta\tan^2\theta}{\cos\theta}$

2 Write down the value(s) of $\cot x$ in each of the following equations:

a $5\sin x = 4\cos x$ **b** $\tan x = -2$ **c** $3\dfrac{\sin x}{\cos x} = \dfrac{\cos x}{\sin x}$

3 Using the definitions of **sec**, **cosec**, **cot** and **tan** simplify the following expressions:

a $\sin\theta\cot\theta$ **b** $\tan\theta\cot\theta$

c $\tan 2\theta\operatorname{cosec}2\theta$ **d** $\cos\theta\sin\theta(\cot\theta+\tan\theta)$

e $\sin^3 x\operatorname{cosec}x + \cos^3 x\sec x$ **f** $\sec A - \sec A\sin^2 A$

g $\sec^2 x\cos^5 x + \cot x\operatorname{cosec}x\sin^4 x$

4 Show that

a $\cos\theta + \sin\theta\tan\theta \equiv \sec\theta$ **b** $\cot\theta + \tan\theta \equiv \operatorname{cosec}\theta\sec\theta$

c $\operatorname{cosec}\theta - \sin\theta \equiv \cos\theta\cot\theta$ **d** $(1-\cos x)(1+\sec x) \equiv \sin x\tan x$

e $\dfrac{\cos x}{1-\sin x} + \dfrac{1-\sin x}{\cos x} \equiv 2\sec x$ **f** $\dfrac{\cos\theta}{1+\cot\theta} \equiv \dfrac{\sin\theta}{1+\tan\theta}$

5 Solve, for values of θ in the interval $0 \leqslant \theta \leqslant 360°$, the following equations. Give your answers to 3 significant figures where necessary.

a $\sec\theta = \sqrt{2}$ **b** $\operatorname{cosec}\theta = -3$ **c** $5\cot\theta = -2$ **d** $\operatorname{cosec}\theta = 2$

e $3\sec^2\theta - 4 = 0$ **f** $5\cos\theta = 3\cot\theta$ **g** $\cot^2\theta - 8\tan\theta = 0$ **h** $2\sin\theta = \operatorname{cosec}\theta$

6 Solve, for values of θ in the interval $-180° \leqslant \theta \leqslant 180°$, the following equations:

a $\operatorname{cosec}\theta = 1$ **b** $\sec\theta = -3$

c $\cot\theta = 3.45$ **d** $2\operatorname{cosec}^2\theta - 3\operatorname{cosec}\theta = 0$

e $\sec\theta = 2\cos\theta$ **f** $3\cot\theta = 2\sin\theta$

g $\operatorname{cosec}2\theta = 4$ **h** $2\cot^2\theta - \cot\theta - 5 = 0$

7 Solve the following equations for values of θ in the interval $0 \leqslant \theta \leqslant 2\pi$. Give your answers in terms of π.

a $\sec\theta = -1$ **b** $\cot\theta = -\sqrt{3}$

c $\operatorname{cosec}\frac{1}{2}\theta = \dfrac{2\sqrt{3}}{3}$ **d** $\sec\theta = \sqrt{2}\tan\theta \left(\theta \neq \dfrac{\pi}{2}, \theta \neq \dfrac{3\pi}{2}\right)$

Exercise B

Give answers to 3 significant figures where necessary.

1 Simplify each of the following expressions:

a $1 + \tan^2 \frac{1}{2}\theta$
b $(\sec\theta - 1)(\sec\theta + 1)$
c $\tan^2\theta(\operatorname{cosec}^2\theta - 1)$
d $(\sec^2\theta - 1)\cot\theta$
e $(\operatorname{cosec}^2\theta - \cot^2\theta)^2$
f $2 - \tan^2\theta + \sec^2\theta$
g $\dfrac{\tan\theta\sec\theta}{1+\tan^2\theta}$
h $(1 - \sin^2\theta)(1 + \tan^2\theta)$
i $\dfrac{\operatorname{cosec}\theta\cot\theta}{1+\cot^2\theta}$
j $(\sec^4\theta - 2\sec^2\theta\tan^2\theta + \tan^4\theta)$
k $4\operatorname{cosec}^2 2\theta + 4\operatorname{cosec}^2 2\theta\cot^2 2\theta$

2 Given that $\operatorname{cosec} x = \dfrac{k}{\operatorname{cosec} x}$, where $k > 1$, find, in terms of k, possible values of $\cot x$.

3 Given that $\cot\theta = -\sqrt{3}$, and that $90° < \theta < 180°$, find the exact value of

a $\sin\theta$
b $\cos\theta$

4 Given that $\tan\theta = \frac{3}{4}$, and that $180° < \theta < 270°$, find the exact value of

a $\sec\theta$
b $\cos\theta$
c $\sin\theta$

5 Given that $\cos\theta = \frac{24}{25}$, and that θ is a reflex angle, find the exact value of

a $\tan\theta$
b $\operatorname{cosec}\theta$

6 Prove the following identities:

a $\sec^4\theta - \tan^4\theta \equiv \sec^2\theta + \tan^2\theta$
b $\operatorname{cosec}^2 x - \sin^2 x \equiv \cot^2 x + \cos^2 x$
c $\sec^2 A(\cot^2 A - \cos^2 A) \equiv \cot^2 A$
d $1 - \cos^2\theta \equiv (\sec^2\theta - 1)(1 - \sin^2\theta)$
e $\dfrac{1-\tan^2 A}{1+\tan^2 A} \equiv 1 - 2\sin^2 A$
f $\sec^2\theta + \operatorname{cosec}^2\theta \equiv \sec^2\theta\operatorname{cosec}^2\theta$
g $\operatorname{cosec} A\sec^2 A \equiv \operatorname{cosec} A + \tan A\sec A$
h $(\sec\theta - \sin\theta)(\sec\theta + \sin\theta) \equiv \tan^2\theta + \cos^2\theta$

7 Given that $3\tan^2\theta + 4\sec^2\theta = 5$, and that θ is obtuse, find the exact value of $\sin\theta$.

8 Solve the following equations in the given intervals:

a $\sec^2\theta = 3\tan\theta,\ 0 \leqslant \theta \leqslant 360°$
b $\tan^2\theta - 2\sec\theta + 1 = 0,\ -\pi \leqslant \theta \leqslant \pi$
c $\operatorname{cosec}^2\theta + 1 = 3\cot\theta,\ -180° \leqslant \theta \leqslant 180°$
d $\cot\theta = 1 - \operatorname{cosec}^2\theta,\ 0 \leqslant \theta \leqslant 2\pi$
e $3\sec\frac{1}{2}\theta = 2\tan^2\frac{1}{2}\theta,\ 0 \leqslant \theta \leqslant 360°$
f $(\sec\theta - \cos\theta)^2 = \tan\theta - \sin^2\theta,\ 0 \leqslant \theta \leqslant \pi$
g $\tan^2 2\theta = \sec 2\theta - 1,\ 0 \leqslant \theta \leqslant 180°$
h $\sec^2\theta - (1+\sqrt{3})\tan\theta + \sqrt{3} = 1,\ 0 \leqslant \theta \leqslant 2\pi$

9 Given that $\tan^2 k = 2\sec k$,

a find the value of $\sec k$
b deduce that $\cos k = \sqrt{2} - 1$
c hence solve, in the interval $0 \leqslant k \leqslant 360°$, $\tan^2 k = 2\sec k$, giving your answers to 1 decimal place.

10 Given that $a = 4\sec x$, $b = \cos x$ and $c = \cot x$,

a express b in terms of a
b show that $c^2 = \dfrac{16}{a^2 - 16}$

11 Given that $x = \sec\theta + \tan\theta$,

a show that $\frac{1}{x} = \sec\theta - \tan\theta$.

b Hence express $x^2 + \frac{1}{x^2} + 2$ in terms of θ, in its simplest form.

12 Given that $2\sec^2\theta - \tan^2\theta = p$ show that $\operatorname{cosec}^2\theta = \frac{p-1}{p-2}$, $p \neq 2$.

Exercise C

1 A student makes the mistake of thinking that $\sin(A + B) \equiv \sin A + \sin B$.
Choose non-zero values of A and B to show that this statement is not true for all values of A and B.

This is a very common error – don't make the same mistake. One counterexample is sufficient to disprove a statement.

2 Using the expansion of $\cos(A - B)$ with $A = B = \theta$, show that $\sin^2\theta + \cos^2\theta \equiv 1$.

3 **a** Use the expansion of $\sin(A - B)$ to show that $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$.

b Use the expansion of $\cos(A - B)$ to show that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

4 Express the following as a single sine, cosine or tangent:

a $\sin 15° \cos 20° + \cos 15° \sin 20°$

b $\sin 58° \cos 23° - \cos 58° \sin 23°$

c $\cos 130° \cos 80° - \sin 130° \sin 80°$

d $\frac{\tan 76° - \tan 45°}{1 + \tan 76° \tan 45°}$

e $\cos 2\theta \cos\theta + \sin 2\theta \sin\theta$

f $\cos 4\theta \cos 3\theta - \sin 4\theta \sin 3\theta$

g $\sin\frac{1}{2}\theta \cos 2\frac{1}{2}\theta + \cos\frac{1}{2}\theta \sin 2\frac{1}{2}\theta$

h $\frac{\tan 2\theta + \tan 3\theta}{1 - \tan 2\theta \tan 3\theta}$

i $\sin(A + B)\cos B - \cos(A + B)\sin B$

j $\cos\left(\frac{3x+2y}{2}\right)\cos\left(\frac{3x-2y}{2}\right) - \sin\left(\frac{3x+2y}{2}\right)\sin\left(\frac{3x-2y}{2}\right)$

5 Calculate, without using your calculator, the exact value of:

a $\sin 30° \cos 60° + \cos 30° \sin 60°$

b $\cos 110° \cos 20° + \sin 110° \sin 20°$

c $\sin 33° \cos 27° + \cos 33° \sin 27°$

d $\cos\frac{\pi}{8}\cos\frac{\pi}{8} - \sin\frac{\pi}{8}\sin\frac{\pi}{8}$

e $\sin 60° \cos 15° - \cos 60° \sin 15°$

f $\cos 70°(\cos 50° - \tan 70° \sin 50°)$

g $\frac{\tan 45° + \tan 15°}{1 - \tan 45° \tan 15°}$

h $\frac{1 - \tan 15°}{1 + \tan 15°}$

Hint: $\tan 45° = 1$.

i $\frac{\tan\left(\frac{7\pi}{12}\right) - \tan\left(\frac{\pi}{3}\right)}{1 + \tan\left(\frac{7\pi}{12}\right)\tan\left(\frac{\pi}{3}\right)}$

j $\sqrt{3}\cos 15° - \sin 15°$

Hint: Look at **e**.

6 Triangle ABC is such that $AB = 3$ cm, $BC = 4$ cm, $\angle ABC = 120°$ and $\angle BAC = \theta°$.

a Write down, in terms of θ, an expression for $\angle ACB$.

b Using the sine rule, or otherwise, show that $\tan\theta° = \frac{2\sqrt{3}}{5}$.

7 Prove the identities

a $\sin(A + 60°) + \sin(A - 60°) \equiv \sin A$

b $\dfrac{\cos A}{\sin B} - \dfrac{\sin A}{\cos B} \equiv \dfrac{\cos(A + B)}{\sin B \cos B}$

c $\dfrac{\sin(x + y)}{\cos x \cos y} \equiv \tan x + \tan y$

d $\dfrac{\cos(x + y)}{\sin x \sin y} + 1 \equiv \cot x \cot y$

e $\cos\left(\theta + \dfrac{\pi}{3}\right) + \sqrt{3}\sin\theta \equiv \sin\left(\theta + \dfrac{\pi}{6}\right)$

f $\cot(A + B) \equiv \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$

g $\sin^2(45 + \theta)° + \sin^2(45 - \theta)° \equiv 1$

h $\cos(A + B)\cos(A - B) \equiv \cos^2 A - \sin^2 B$

8 Given that $\sin A = \frac{4}{5}$ and $\sin B = \frac{1}{2}$, where A and B are both acute angles, calculate the exact values of

a $\sin(A + B)$ **b** $\cos(A - B)$ **c** $\sec(A - B)$

9 Given that $\cos A = -\frac{4}{5}$, and A is an obtuse angle measured in radians, find the exact value of

a $\sin A$ **b** $\cos(\pi + A)$ **c** $\sin\left(\dfrac{\pi}{3} + A\right)$ **d** $\tan\left(\dfrac{\pi}{4} + A\right)$

10 Given that $\sin A = \frac{8}{17}$, where A is acute, and $\cos B = -\frac{4}{5}$, where B is obtuse, calculate the exact value of

a $\sin(A - B)$ **b** $\cos(A - B)$ **c** $\cot(A - B)$

11 Given that $\tan A = \frac{7}{24}$, where A is reflex, and $\sin B = \frac{5}{13}$, where B is obtuse, calculate the exact value of

a $\sin(A + B)$ **b** $\tan(A - B)$ **c** $\operatorname{cosec}(A + B)$

12 Write the following as a single trigonometric function, assuming that θ is measured in radians:

a $\cos^2\theta - \sin^2\theta$ **b** $2\sin 4\theta \cos 4\theta$ **c** $\dfrac{1 + \tan\theta}{1 - \tan\theta}$ **d** $\dfrac{1}{\sqrt{2}}(\sin\theta + \cos\theta)$

13 Solve, in the interval $0° \leqslant \theta < 360°$, the following equations. Give answers to the nearest 0.1°.

a $3\cos\theta = 2\sin(\theta + 60°)$

b $\sin(\theta + 30°) + 2\sin\theta = 0$

c $\cos(\theta + 25°) + \sin(\theta + 65°) = 1$

d $\cos\theta = \cos(\theta + 60°)$

e $\tan(\theta - 45°) = 6\tan\theta$

f $\sin\theta + \cos\theta = 1$

Hint for part **f**: Multiply each term by $\dfrac{1}{\sqrt{2}}$

14 **a** Solve the equation $\cos\theta\cos 30° - \sin\theta\sin 30° = 0.5$, for $0 \leqslant \theta \leqslant 360°$.

b Hence write down, in the same interval, the solutions of $\sqrt{3}\cos\theta - \sin\theta = 1$.

15 **a** Express $\tan(45 + 30)°$ in terms of $\tan 45°$ and $\tan 30°$.

b Hence show that $\tan 75° = 2 + \sqrt{3}$.

16 Show that $\sec 105° = -\sqrt{2}(1 + \sqrt{3})$

17 Calculate the exact values of

a $\cos 15°$

b $\sin 75°$

c $\sin(120 + 45)°$

d $\tan 165°$

Hint for part **a:** Write 15° as $(45 - 30)°$

18 **a** Given that $3\sin(x - y) - \sin(x + y) = 0$, show that $\tan x = 2\tan y$.

b Solve $3\sin(x - 45°) - \sin(x + 45°) = 0$, for $0 \leqslant \theta \leqslant 360°$.

19 Given that $\sin x(\cos y + 2\sin y) = \cos x(2\cos y - \sin y)$, find the value of $\tan(x + y)$.

20 Given that $\tan(x - y) = 3$, express $\tan y$ in terms of $\tan x$.

21 In each of the following, calculate the exact value of $\tan x°$.

a $\tan(x-45)° = \frac{1}{4}$

b $\sin(x-60)° = 3\cos(x+30)°$

c $\tan(x-60)° = 2$

22 Given that $\tan A° = \frac{1}{5}$ and $\tan B° = \frac{2}{3}$, calculate, without using your calculator, the value of $A+B$,

a where A and B are both acute,

b where A is reflex and B is acute.

23 Given that $\cos y = \sin(x+y)$, show that $\tan y = \sec x - \tan x$.

24 Given that $\cot A = \frac{1}{4}$ and $\cot(A+B) = 2$, find the value of $\cot B$.

25 Given that $\tan\left(x+\frac{\pi}{3}\right) = \frac{1}{2}$, show that $\tan x = 8-5\sqrt{3}$.

Exercise D

In equations, give answers to 1 decimal place where appropriate.

1 Write the following expressions as a single trigonometric ratio:

a $2\sin 10°\cos 10°$

b $1-2\sin^2 25°$

c $\cos^2 40° - \sin^2 40°$

d $\dfrac{2\tan 5°}{1-\tan^2 5°}$

e $\dfrac{1}{2\sin(24\frac{1}{2})°\cos(24\frac{1}{2})°}$

f $6\cos^2 30° - 3$

g $\dfrac{\sin 8°}{\sec 8°}$

h $\cos^2\dfrac{\pi}{16} - \sin^2\dfrac{\pi}{16}$

2 Without using your calculator find the exact values of:

a $2\sin(22\frac{1}{2})°\cos(22\frac{1}{2})°$

b $2\cos^2 15° - 1$

c $(\sin 75° - \cos 75°)^2$

d $\dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

3 Write the following in their simplest form, involving only one trigonometric function:

a $\cos^2 3\theta - \sin^2 3\theta$

b $6\sin 2\theta\cos 2\theta$

c $\dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

d $2-4\sin^2\dfrac{\theta}{2}$

e $\sqrt{1+\cos 2\theta}$

f $\sin^2\theta\cos^2\theta$

g $4\sin\theta\cos\theta\cos 2\theta$

h $\dfrac{\tan\theta}{\sec^2\theta - 2}$

i $\sin^4\theta - 2\sin^2\theta\cos^2\theta + \cos^4\theta$

4 Given that $\cos x = \frac{1}{4}$, find the exact value of $\cos 2x$.

5 Find the possible values of $\sin\theta$ when $\cos 2\theta = \frac{23}{25}$.

6 Given that $\cos x + \sin x = m$ and $\cos x - \sin x = n$, where m and n are constants, write down, in terms of m and n, the value of $\cos 2x$.

7 Given that $\tan\theta = \frac{3}{4}$, and that θ is acute:

a Find the exact value of **i** $\tan 2\theta$ **ii** $\sin 2\theta$ **iii** $\cos 2\theta$

b Deduce the value of $\sin 4\theta$.

8 Given that $\cos A = -\frac{1}{3}$, and that A is obtuse:

a Find the exact value of **i** $\cos 2A$ **ii** $\sin A$ **iii** $\operatorname{cosec} 2A$

b Show that $\tan 2A = \dfrac{4\sqrt{2}}{7}$.

9 Given that $\pi < \theta < \dfrac{3\pi}{2}$, find the value of $\tan\dfrac{\theta}{2}$ when $\tan\theta = \frac{3}{4}$.

10 In $\triangle ABC$, $AB = 4$ cm, $AC = 5$ cm, $\angle ABC = 2\theta$ and $\angle ACB = \theta$. Find the value of θ, giving your answer, in degrees, to 1 decimal place.

11 In $\triangle PQR$, $PQ = 3$ cm, $PR = 6$ cm, $QR = 5$ cm and $\angle QPR = 2\theta$.

a Use the cosine rule to show that $\cos 2\theta = \frac{5}{9}$.

b Hence find the exact value of $\sin\theta$.

12 The line l, with equation $y = \frac{3}{4}x$, bisects the angle between the x-axis and the line $y = mx$, $m > 0$. Given that the scales on each axis are the same, and that l makes an angle θ with the x-axis,

a write down the value of $\tan\theta$.

b Show that $m = \frac{24}{7}$.

Exercise E

In equations, give answers to 1 decimal place where appropriate.

1 Prove the following identities:

a $\dfrac{\cos 2A}{\cos A + \sin A} \equiv \cos A - \sin A$

b $\dfrac{\sin B}{\sin A} - \dfrac{\cos B}{\cos A} \equiv 2\operatorname{cosec} 2A \sin(B - A)$

c $\dfrac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan\theta$

d $\dfrac{\sec^2\theta}{1 - \tan^2\theta} \equiv \sec 2\theta$

e $2(\sin^3\theta\cos\theta + \cos^3\theta\sin\theta) \equiv \sin 2\theta$

f $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} \equiv 2$

g $\operatorname{cosec}\theta - 2\cot 2\theta\cos\theta \equiv 2\sin\theta$

h $\dfrac{\sec\theta - 1}{\sec\theta + 1} \equiv \tan^2\dfrac{\theta}{2}$

i $\tan\left(\dfrac{\pi}{4} - x\right) \equiv \dfrac{1 - \sin 2x}{\cos 2x}$

2 **a** Show that $\tan\theta + \cot\theta \equiv 2\operatorname{cosec} 2\theta$.

b Hence find the value of $\tan 75° + \cot 75°$.

3 Solve the following equations, in the interval shown in brackets:

a $\sin 2\theta = \sin\theta \quad \{0 \leqslant \theta \leqslant 2\pi\}$

b $\cos 2\theta = 1 - \cos\theta \quad \{-180° < \theta \leqslant 180°\}$

c $3\cos 2\theta = 2\cos^2\theta \quad \{0 \leqslant \theta < 360°\}$

d $\sin 4\theta = \cos 2\theta \quad \{0 \leqslant \theta \leqslant \pi\}$

e $2\tan 2y \tan y = 3 \quad \{0 \leqslant y < 360°\}$

f $3\cos\theta - \sin\dfrac{\theta}{2} - 1 = 0 \quad \{0 \leqslant \theta < 720°\}$

g $\cos^2\theta - \sin 2\theta = \sin^2\theta \quad \{0 \leqslant \theta \leqslant \pi\}$

h $2\sin\theta = \sec\theta \quad \{0 \leqslant \theta \leqslant 2\pi\}$

i $2\sin 2\theta = 3\tan\theta \quad \{0 \leqslant \theta < 360°\}$

j $2\tan\theta = \sqrt{3}(1 - \tan\theta)(1 + \tan\theta) \quad \{0 \leqslant \theta \leqslant 2\pi\}$

k $5\sin 2\theta + 4\sin\theta = 0 \quad \{-180° < \theta \leqslant 180°\}$

l $\sin^2\theta = 2\sin 2\theta \quad \{-180° < \theta \leqslant 180°\}$

m $4\tan\theta = \tan 2\theta \quad \{0 \leqslant \theta < 360°\}$

4 Given that $p = 2\cos\theta$ and $q = \cos 2\theta$, express q in terms of p.

5 Eliminate θ from the following pairs of equations:

a $x = \cos^2\theta,\ y = 1 - \cos 2\theta$ **b** $x = \tan\theta,\ y = \cot 2\theta$

c $x = \sin\theta,\ y = \sin 2\theta$ **d** $x = 3\cos 2\theta + 1,\ y = 2\sin\theta$

6 **a** Prove that $(\cos 2\theta - \sin 2\theta)^2 \equiv 1 - \sin 4\theta$.

b Use the result to solve, for $0 \leqslant \theta < \pi$, the equation $\cos 2\theta - \sin 2\theta = \dfrac{1}{\sqrt{2}}$.
Give your answers in terms of π.

7 **a** Show that:

i $\sin\theta \equiv \dfrac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$ **ii** $\cos\theta \equiv \dfrac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$

b By writing the following equations as quadratics in $\tan\dfrac{\theta}{2}$, solve, in the interval $0 \leqslant \theta \leqslant 360°$:

i $\sin\theta + 2\cos\theta = 1$ **ii** $3\cos\theta - 4\sin\theta = 2$

8 **a** Using $\cos 2A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$, show that:

i $\cos^2\dfrac{x}{2} \equiv \dfrac{1 + \cos x}{2}$ **ii** $\sin^2\dfrac{x}{2} \equiv \dfrac{1 - \cos x}{2}$

These are known as the **half angle formulae**. (They are useful in integration.)

b Given that $\cos\theta = 0.6$, and that θ is acute, write down the values of:

i $\cos\dfrac{\theta}{2}$ **ii** $\sin\dfrac{\theta}{2}$ **iii** $\tan\dfrac{\theta}{2}$

c Show that $\cos^4\dfrac{A}{2} \equiv \frac{1}{8}(3 + 4\cos A + \cos 2A)$

9 **a** Show that $3\cos^2 x - \sin^2 x \equiv 1 + 2\cos 2x$.

b Hence sketch, for $-\pi \leqslant x \leqslant \pi$, the graph of $y = 3\cos^2 x - \sin^2 x$, showing the coordinates of points where the curve meets the axes.

10 **a** Express $2\cos^2\dfrac{\theta}{2} - 4\sin^2\dfrac{\theta}{2}$ in the form $a\cos\theta + b$, where a and b are constants.

b Hence solve $2\cos^2\dfrac{\theta}{2} - 4\sin^2\dfrac{\theta}{2} = -3$, in the interval $0 \leqslant \theta < 360°$.

11 **a** Use the identity $\sin^2 A + \cos^2 A \equiv 1$ to show that $\sin^4 A + \cos^4 A \equiv \frac{1}{2}(2 - \sin^2 2A)$.

b Deduce that $\sin^4 A + \cos^4 A \equiv \frac{1}{4}(3 + \cos 4A)$.

c Hence solve $8\sin^4\theta + 8\cos^4\theta = 7$, for $0 < \theta < \pi$.

12 **a** By expanding $\cos(2A + A)$ show that $\cos 3A \equiv 4\cos^3 A - 3\cos A$.

b Hence solve $8\cos^3\theta - 6\cos\theta - 1 = 0$, for $\{0 \leqslant \theta \leqslant 360°\}$.

13 **a** Show that $\tan 3\theta \equiv \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

b Given that θ is acute such that $\cos\theta = \frac{1}{3}$, show that $\tan 3\theta = \dfrac{10\sqrt{2}}{23}$.

Hint: Use formulae for $\sin 3\theta$ and $\cos 3\theta$. See Example 13 and question **12** above.

Exercise F

Give all angles to the nearest 0.1° and non-exact values of R in surd form.

1 Given that $5\sin\theta + 12\cos\theta \equiv R\sin(\theta + \alpha)$, find the value of R; $R > 0$, and the value of $\tan\alpha$.

2 Given that $\sqrt{3}\sin\theta + \sqrt{6}\cos\theta \equiv 3\cos(\theta - \alpha)$, where $0 < \alpha < 90°$, find the value of α.

3 Given that $2\sin\theta - \sqrt{5}\cos\theta \equiv -3\cos(\theta + \alpha)$, where $0 < \alpha < 90°$, find the value of α.

4 Show that:

a $\cos\theta + \sin\theta \equiv \sqrt{2}\sin\left(\theta + \dfrac{\pi}{4}\right)$

b $\sqrt{3}\sin 2\theta - \cos 2\theta \equiv 2\sin\left(2\theta - \dfrac{\pi}{6}\right)$

5 Prove that $\cos 2\theta - \sqrt{3}\sin 2\theta \equiv 2\cos\left(2\theta + \dfrac{\pi}{3}\right) \equiv -2\sin\left(2\theta - \dfrac{\pi}{6}\right)$.

6 Find the value of R, where $R > 0$, and the value of α, where $0 < \alpha < 90°$, in each of the following cases:

a $\sin\theta + 3\cos\theta \equiv R\sin(\theta + \alpha)$

b $3\sin\theta - 4\cos\theta \equiv R\sin(\theta - \alpha)$

c $2\cos\theta + 7\sin\theta \equiv R\cos(\theta - \alpha)$

d $\cos 2\theta - 2\sin 2\theta \equiv R\cos(2\theta + \alpha)$

7 **a** Show that $\cos\theta - \sqrt{3}\sin\theta$ can be written in the form $R\cos(\theta + \alpha)$, with $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$.

b Hence sketch the graph of $y = \cos\theta - \sqrt{3}\sin\theta$, $0 < \alpha < 2\pi$, giving the coordinates of points of intersection with the axes.

8 **a** Show that $3\sin 3\theta - 4\cos 3\theta$ can be written in the form $R\sin(3\theta - \alpha)$, with $R > 0$ and $0 < \alpha < 90°$.

b Deduce the minimum value of $3\sin 3\theta - 4\cos 3\theta$ and work out the smallest positive value of θ at which it occurs.

9 **a** Show that $\cos 2\theta + \sin 2\theta$ can be written in the form $R\sin(2\theta + \alpha)$, with $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

b Hence solve, in the interval $0 \leqslant \theta < 2\pi$, the equation $\cos 2\theta + \sin 2\theta = 1$, giving your answers as rational multiples of π.

10 **a** Express $7\cos\theta - 24\sin\theta$ in the form $R\cos(\theta + \alpha)$, with $R > 0$ and $0 < \alpha < 90°$.

b The graph of $y = 7\cos\theta - 24\sin\theta$ meets the y-axis at P. State the coordinates of P.

c Write down the maximum and minimum values of $7\cos\theta - 24\sin\theta$.

d Deduce the number of solutions, in the interval $0 < \theta < 360°$, of the following equations:

i $7\cos\theta - 24\sin\theta = 15$ **ii** $7\cos\theta - 24\sin\theta = 26$ **iii** $7\cos\theta - 24\sin\theta = -25$

11 **a** Express $5\sin^2\theta - 3\cos^2\theta + 6\sin\theta\cos\theta$ in the form $a\sin 2\theta + b\cos 2\theta + c$, where a, b and c are constants.

b Hence find the maximum and minimum values of $5\sin^2\theta - 3\cos^2\theta + 6\sin\theta\cos\theta$.

12 Solve the following equations, in the interval given in brackets:

a $6\sin x + 8\cos x = 5\sqrt{3}$ $[0, 360°]$

b $2\cos 3\theta - 3\sin 3\theta = -1$ $[0, 90°]$

c $8\cos\theta + 15\sin\theta = 10$ $[0, 360°]$

d $5\sin\frac{x}{2} - 12\cos\frac{x}{2} = -6.5$ $[-360°, 360°]$

13 Solve the following equations, in the interval given in brackets:

a $\sin x\cos x = 1 - 2.5\cos 2x$ $[0, 360°]$

b $\cot\theta + 2 = \operatorname{cosec}\theta$ $[0, 360°]$

c $\sin\theta = 2\cos\theta - \sec\theta$ $[0, 180°]$

d $\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) + (\sqrt{3} - 1)\sin\theta = 2$ $[0, 2\pi]$

14 Solve, if possible, in the interval $0 < \theta < 360°$, $\theta \neq 180°$, the equation $\dfrac{4 - 2\sqrt{2}\sin\theta}{1 + \cos\theta} = k$ in the case when k is equal to:

a 4 **b** 2 **c** 1 **d** 0 **e** -1

15 A class were asked to solve $3\cos\theta = 2 - \sin\theta$ for $0 \leqslant \theta \leqslant 360°$. One student expressed the equation in the form $R\cos(\theta - \alpha) = 2$, with $R > 0$ and $0 < \alpha < 90°$, and correctly solved the equation.

a Find the values of R and α and hence find her solutions.

Another student decided to square both sides of the equation and then form a quadratic equation in $\sin\theta$.

b Show that the correct quadratic equation is $10\sin^2\theta - 4\sin\theta - 5 = 0$.

c Solve this equation, for $0 \leqslant \theta < 360°$.

d Explain why not all of the answers satisfy $3\cos\theta = 2 - \sin\theta$.