Trigonometry

Convention for representing angles

In the A-Level Trigonometry, in addition to working with angles smaller than 360° we will also work with angles which are bigger than 360° as well as negative angles. We follow the following conventions when working with these angles:

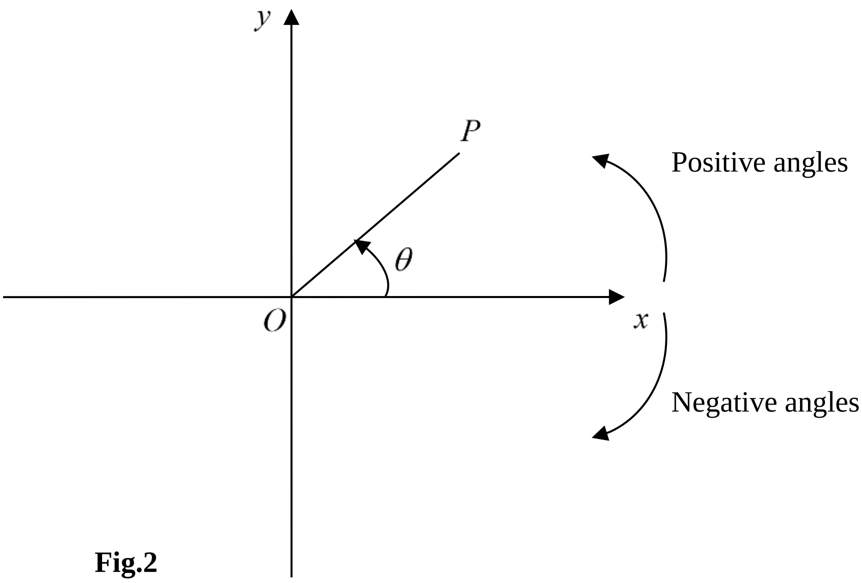

Fig.2

- Angles are measured from the positive *x*-axis to a line OP.
- Angles measured in the anti-clockwise sense are positive.
- Angles measured in the clockwise sense are negative.
- Angles bigger than 360° or angles smaller than -360°, mean more than one full rotation from the positive *x*-axis.

Some useful angles

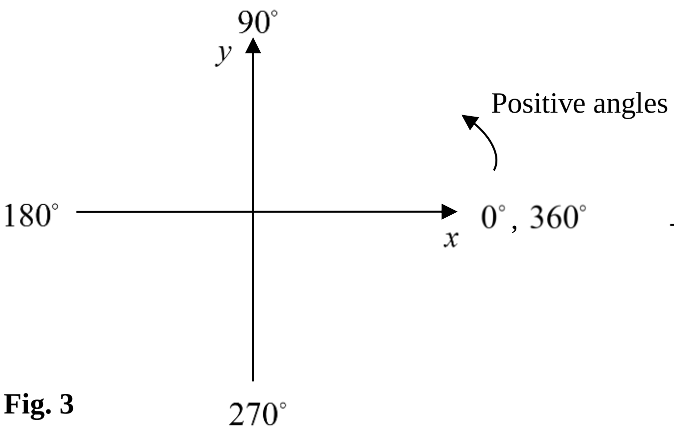

Fig. 3

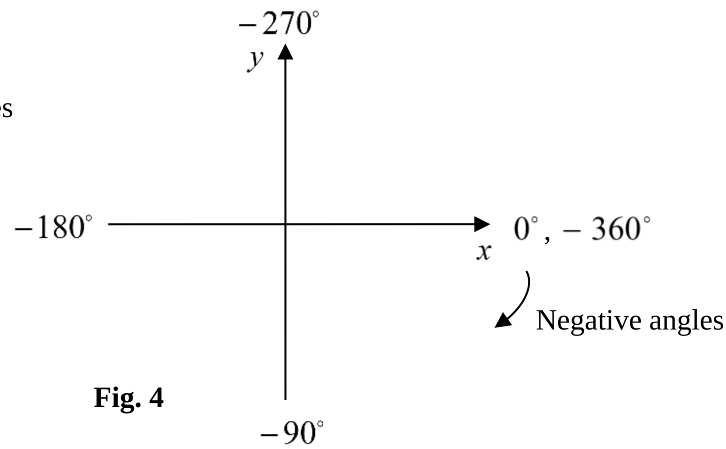

Fig. 4

Exercise 1

Draw separate diagrams to show each of the following angles in an *x*-*y* plane.

a) 35° b) 340° c) 240° d) 150° e) -250° f) -340°

g) 420° h) 570° i) -600°

Definitions of Sine, Cosine and Tangent for any angle

In the GCSEs you used Sine, Cosine and Tangent mostly with angles smaller than 90°. However these trigonometric ratios can actually be used with angles bigger than 90° as well as negative angles too.

The definitions you learnt in the GCSEs for Sine, Cosine and Tangent are only simplified versions of a set of broader definitions. The simplified definitions such as $\sin\theta = \frac{opp}{hyp}$ cannot be used with angles bigger than 90° or negative angles, as no right-angled triangle can be drawn with such angles. The broader definitions for these trigonometric ratios, which are given below, allow us to use them with any angle irrespective of its size and sign.

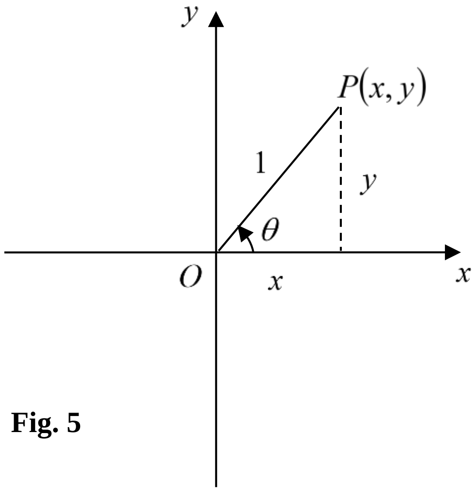

Fig. 5

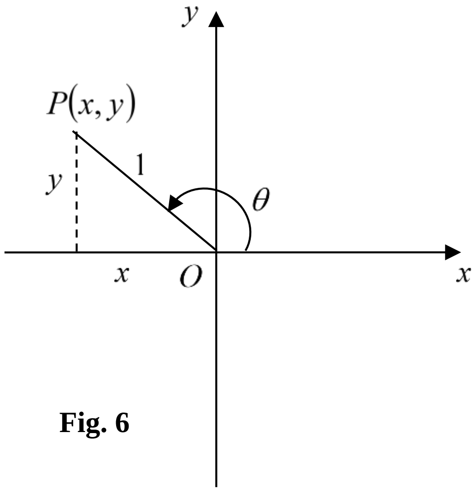

Fig. 6

In the broader definitions, Sine, Cosine and Tangent are defined using a rotating line, OP, of length 1 unit as follows:

$$\sin\theta = \frac{y}{1}$$

$$\cos\theta = \frac{x}{1}$$

$$\tan\theta = \frac{y}{x}$$

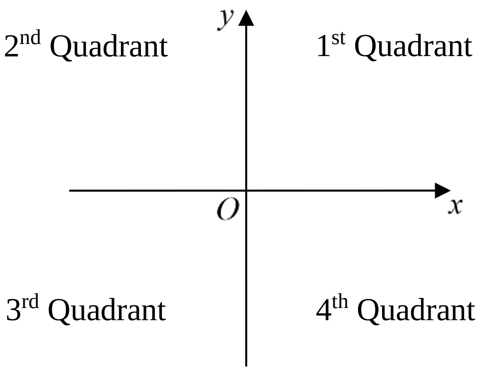

Fig. 7

Depending on the quadrant in which an angle falls, the *x* and *y* coordinates may be positive or negative. The length of OP (1 unit) is considered as a positive quantity irrespective of the quadrant where OP falls. This means, the Sine, Cosine and the Tangent ratios may be positive of negative depending on the quadrant in which a given angle falls.

Examples

1) $\sin 250^\circ$ will be negative because, the angle 250° falls in the 3rd quadrant, where the *y* coordinate is negative.
2) $\cos(-310^\circ)$ will be positive because, the angle -310° falls in the 1st quadrant, where the *x* coordinate is positive.

Exercise 2

State whether each of the following trigonometric ratios is positive or negative.

a) $\sin 140^\circ$ b) $\tan 140^\circ$ c) $\cos 140^\circ$ d) $\sin 340^\circ$
e) $\cos 340^\circ$ f) $\tan 340^\circ$ g) $\sin(-217^\circ)$ h) $\tan(-217^\circ)$
i) $\cos(-740^\circ)$

The CAST Diagram

The CAST diagram summarizes the signs of the Sine, Cosine and Tangent ratios in different quadrants. You will find it very easy to determine the sign of any trigonometric ratio for any angle using the CAST diagram.

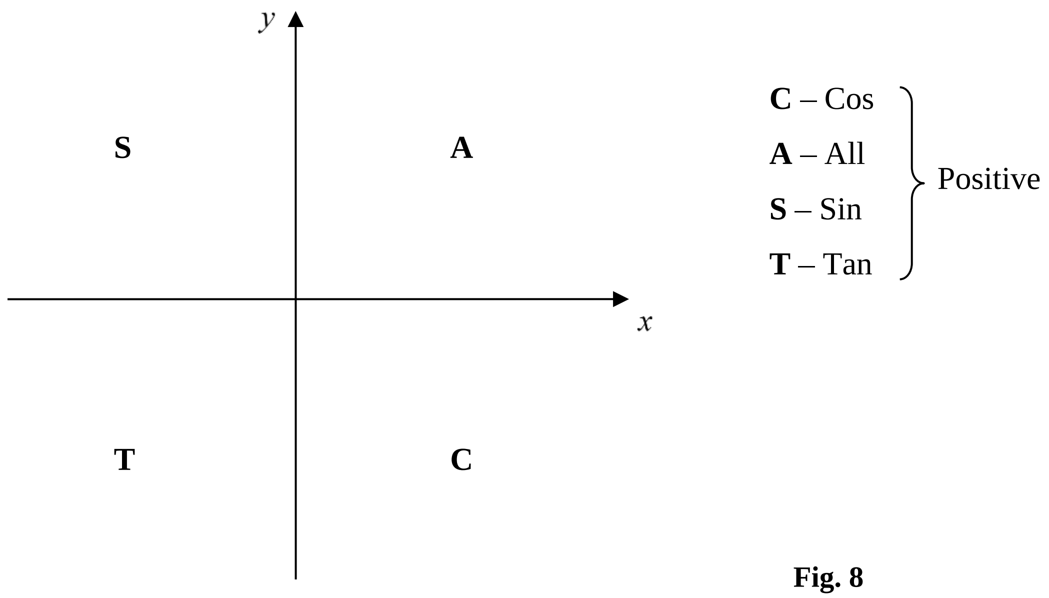

Fig. 8

The **S** in the 2nd quadrant tells that **only** Sin is positive for any angle falling into that quadrant and Cos and Tan are negative. Similar interpretations can be given for the other quadrants too.

Exercise 3

Use the CAST diagram to determine the signs of each of the trigonometric ratios given in Exercise 2 above.

Graphs of Trigonometric Functions

You should be able to sketch the graphs of,

$y = \sin x$,

$y = \cos x$ and

$y = \tan x$

showing where the graphs cross the coordinate axes, the maximum and minimum points and any asymptotes.

The graph of *y = sin x*

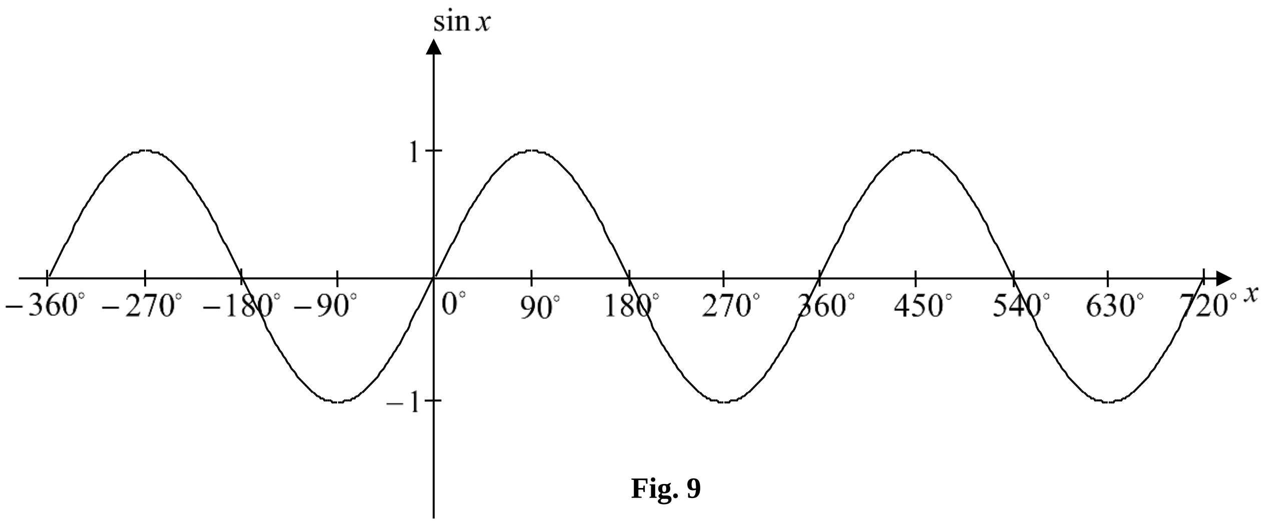

Fig. 9

The graph of *y = cos x*

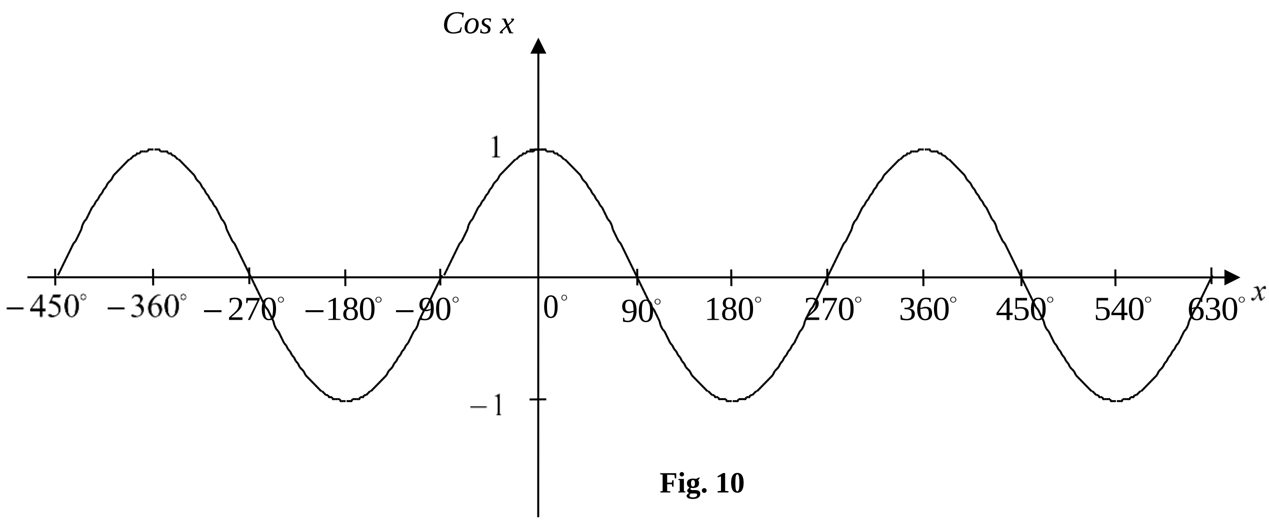

Fig. 10

The Graph of $y = \tan x$

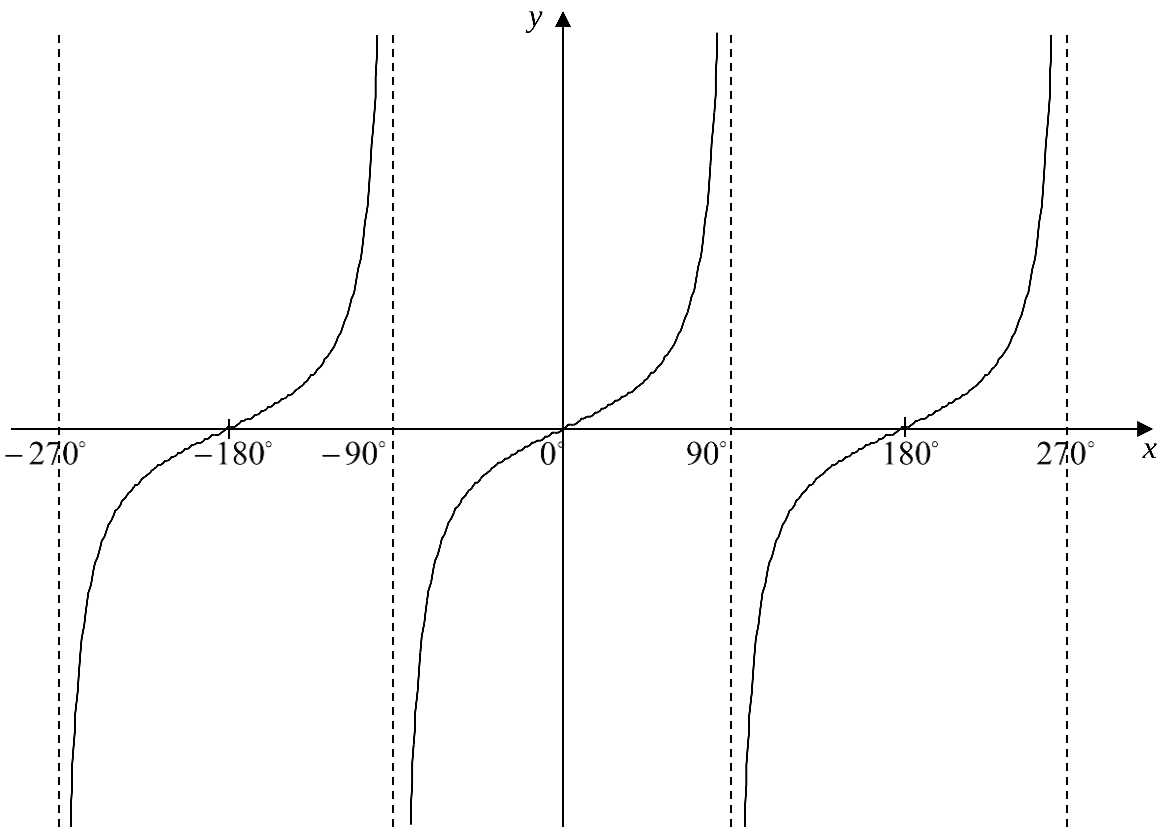

Fig. 11

Periods of Trigonometric Functions

The functions $y = \sin x$, $y = \cos x$ and $y = \tan x$ are all periodic functions.

Function	Period
$y = \sin x$	360°
$y = \cos x$	360°
$y = \tan x$	180°

Maximum and Minimum values of Trigonometric Functions

The maximum possible value of $\sin x$ is 1. This occurs when the angle x is $90^\circ, 450^\circ, 810^\circ ....., -270^\circ, -630^\circ, ... etc$

Exercise 4

Use Fig. 9 to state the minimum possible value of $\sin x$ and a few values of x at which it occurs.

Exercise 5

Use Fig. 10 to state the maximum and minimum possible values of $\cos x$ and a few values of x at which they occur.

Exercise 6

Find the maximum possible values of each of the following functions, stating a few values of x at which it occurs.

1) $3\sin x$
2) $4\cos x$
3) $6+\cos x$
4) $10-\sin x$
5) $\sin(x+30^\circ)$
6) $2\cos(3x-60^\circ)$
7) $\sin 2x+15$
8) $\dfrac{2}{5-\sin x}$
9) $\dfrac{6}{3+2\cos 4x}$

Exercise 7

Find the minimum possible value of each of the trigonometric functions given in Exercise 6 above. State a few possible values of x at which the minimum occurs.

Exercise 8

Find the maximum possible values of each of the following functions. State the least positive value of x at which the maximum occurs.

1) $5\sin x$
2) $3\cos 2x$
3) $4\sin(2x-15^\circ)$
4) $\dfrac{10}{2\sin x-3}$
5) $\dfrac{3}{5-2\cos(3x+45^\circ)}$

Exercise 9

Find the minimum possible value of each of the functions in Exercise 8 above, stating the least positive value at which the minimum occurs.

Exact Values of Trigonometric Functions

You are expected to know the exact values of Sin, Cos and Tan for $0^\circ, 30^\circ, 45^\circ, 60^\circ$ and for all multiples of 90° up to 360°. However, nowadays calculators can give these exact values and therefore you don't really have to learn these by heart.

Symmetric Properties of the Graphs of Trigonometric Functions

The Graphs of the trigonometric functions,

$y = \sin x$,

$y = \cos x$ and

$y = \tan x$

have symmetric properties, which can be used to answer several questions.

Examples

1) Find any four possible values of x other than $x = 20^\circ$, for which $\sin x$ has the same value as $\sin 20^\circ$.

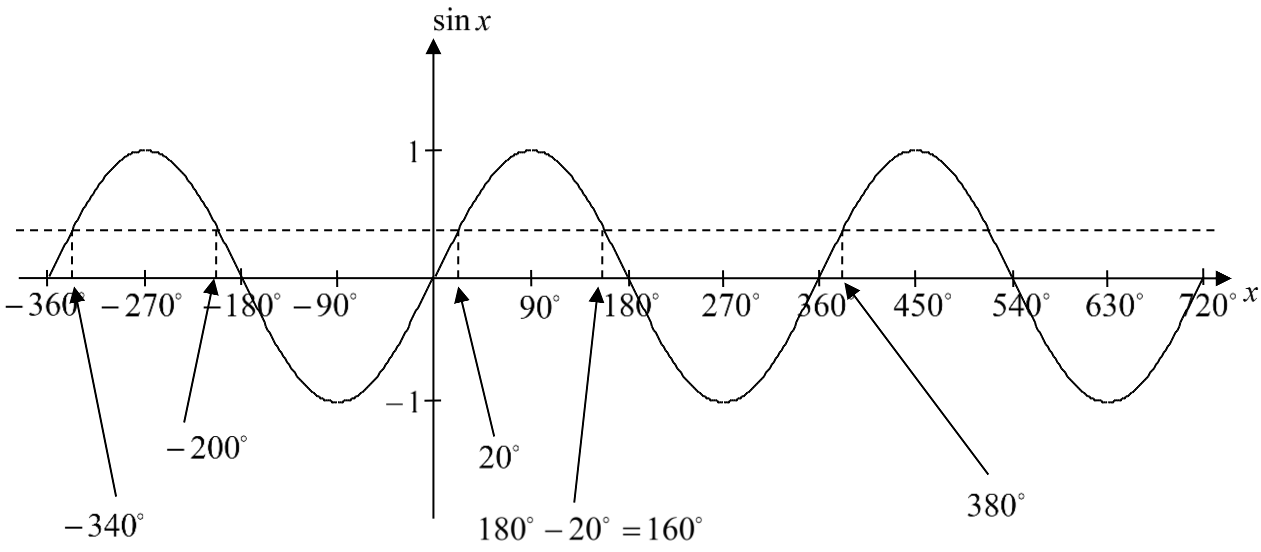

Fig. 12

$$x = (180^\circ - 20^\circ), (360^\circ + 20^\circ), (-180^\circ - 20^\circ), (-360^\circ + 20^\circ).....$$

$$x = 160^\circ, 380^\circ, -200^\circ, -340^\circ.....$$

2) Find all possible values of x, $-360^\circ < x < 360^\circ$, such that $\cos x = \cos 60^\circ$.

This question is similar to the example above, except that you are asked to give only those values of x, which falls in the range, $-360^\circ < x < 360^\circ$. Do this yourself and check whether you get the following answers.

Answers: $x = 60^\circ, 300^\circ, -60^\circ, -300^\circ$.

Solving Trigonometric Equations

There are 3 basic types of trigonometric equations that you should be able to solve. All other types of trigonometric equations should be first simplified into one of these 3 basic types, in order to solve them.

The three basic types

Type 1

Equations such as,

$\sin x = 0.3$, $\quad 0^\circ < x < 360^\circ$

$\cos\theta = -0.7$, $\quad 0^\circ < \theta < 360^\circ$

$\tan x = 2.3$, $\quad -360^\circ \le x < 180^\circ$

Type 2

Equations such as,

$\sin 2\theta = -0.8$, $\quad 0^\circ < \theta < 360^\circ$

$\cos 3x = -0.2$, $\quad -180^\circ < x < 180^\circ$

$\tan 4\beta = 0.9$, $\quad -90^\circ \le \beta < 90^\circ$

Type 3

Equations such as,

$\sin(2\theta + 30^\circ) = -0.8$, $\quad 0^\circ < \theta < 360^\circ$

$\cos(3x - 20^\circ) = -0.2$, $\quad -180^\circ < x < 180^\circ$

$\tan(4\beta - 10^\circ) = 0.9$, $\quad -90^\circ \le \beta < 90^\circ$