Sequences and Series

Exercise A

1 Find the U_1, U_2, U_3 and U_{10} of the following sequences, where:

a $U_n = 3n + 2$
b $U_n = 10 - 3n$
c $U_n = n^2 + 5$
d $U_n = (n - 3)^2$
e $U_n = (-2)^n$
f $U_n = \frac{n}{n+2}$
g $U_n = (-1)^n \frac{n}{n+2}$
h $U_n = (n - 2)^3$

2 Find the value of n for which U_n has the given value:

a $U_n = 2n - 4$, $U_n = 24$
b $U_n = (n - 4)^2$, $U_n = 25$
c $U_n = n^2 - 9$, $U_n = 112$
d $U_n = \frac{2n+1}{n-3}$, $U_n = \frac{19}{6}$
e $U_n = n^2 + 5n - 6$, $U_n = 60$
f $U_n = n^2 - 4n + 11$, $U_n = 56$
g $U_n = n^2 + 4n - 5$, $U_n = 91$
h $U_n = (-1)^n \frac{n}{n+4}$, $U_n = \frac{7}{9}$
i $U_n = \frac{n^3+3}{5}$, $U_n = 13.4$
j $U_n = \frac{n^3}{5} + 3$, $U_n = 28$

3 Prove that the $(2n + 1)$th term of the sequence $U_n = n^2 - 1$ is a multiple of 4.

4 Prove that the terms of the sequence $U_n = n^2 - 10n + 27$ are all positive. For what value of n is U_n smallest?

Hint: Question 4 – Complete the square.

5 A sequence is generated according to the formula $U_n = an + b$, where a and b are constants. Given that $U_3 = 14$ and $U_5 = 38$, find the values of a and b.

6 A sequence is generated according to the formula $U_n = an^2 + bn + c$, where a, b and c are constants. If $U_1 = 4$, $U_2 = 10$ and $U_3 = 18$, find the values of a, b and c.

7 A sequence is generated from the formula $U_n = pn^3 + q$, where p and q are constants. Given that $U_1 = 6$ and $U_3 = 19$, find the values of the constants p and q.

Exercise B

1 Find the first four terms of the following recurrence relationships:

a $U_{n+1} = U_n + 3$, $U_1 = 1$
b $U_{n+1} = U_n - 5$, $U_1 = 9$
c $U_{n+1} = 2U_n$, $U_1 = 3$
d $U_{n+1} = 2U_n + 1$, $U_1 = 2$
e $U_{n+1} = \frac{U_n}{2}$, $U_1 = 10$
f $U_{n+1} = (U_n)^2 - 1$, $U_1 = 2$
g $U_{n+2} = 2U_{n+1} + U_n$, $U_1 = 3$, $U_2 = 5$

2 Suggest possible recurrence relationships for the following sequences (remember to state the first term):

a 3, 5, 7, 9, ...

b 20, 17, 14, 11, ...

c 1, 2, 4, 8, ...

d 100, 25, 6.25, 1.5625, ...

e 1, −1, 1, −1, 1, ...

f 3, 7, 15, 31, ...

g 0, 1, 2, 5, 26, ...

h 26, 14, 8, 5, 3.5, ...

i 1, 1, 2, 3, 5, 8, 13, ...

j 4, 10, 18, 38, 74, ...

3 By writing down the first four terms or otherwise, find the recurrence formula that defines the following sequences:

a $U_n = 2n - 1$

b $U_n = 3n + 2$

c $U_n = n + 2$

d $U_n = \frac{n+1}{2}$

e $U_n = n^2$

f $U_n = (-1)^n n$

4 A sequence of terms $\{U_n\}$ is defined $n \geqslant 1$ by the recurrence relation $U_{n+1} = kU_n + 2$, where k is a constant. Given that $U_1 = 3$:

a find an expression in terms of k for U_2

b hence find an expression for U_3.

Given that $U_3 = 42$:

c find possible values of k.

5 A sequence of terms $\{U_k\}$ is defined $k \geqslant 1$ by the recurrence relation $U_{k+2} = U_{k+1} - pU_k$, where p is a constant. Given that $U_1 = 2$ and $U_2 = 4$:

a find an expression in terms of p for U_3

b hence find an expression in terms of p for U_4.

Given also that U_4 is twice the value of U_3:

c find the value of p.

Exercise C

1 Which of the following sequences are arithmetic?

a 3, 5, 7, 9, 11, ...

b 10, 7, 4, 1, ...

c $y, 2y, 3y, 4y, \ldots$

d 1, 4, 9, 16, 25, ...

e 16, 8, 4, 2, 1, ...

f 1, −1, 1, −1, 1, ...

g $y, y^2, y^3, y^4, \ldots$

h $U_{n+1} = U_n + 2$, $U_1 = 3$

i $U_{n+1} = 3U_n - 2$, $U_1 = 4$

j $U_{n+1} = (U_n)^2$, $U_1 = 2$

k $U_n = n(n+1)$

l $U_n = 2n + 3$

2 Find the 10th and nth terms in the following arithmetic progressions:

a 5, 7, 9, 11, ...

b 5, 8, 11, 14, ...

c 24, 21, 18, 15, ...

d −1, 3, 7, 11, ...

e $x, 2x, 3x, 4x, \ldots$

f $a, a + d, a + 2d, a + 3d, \ldots$

3 An investor puts £4000 in an account. Every month thereafter she deposits another £200. How much money in total will she have invested at the start of **a** the 10th month and **b** the mth month? (Note that at the start of the 6th month she will have made only 5 deposits of £200.)

4 Calculate the number of terms in the following arithmetic sequences:

a 3, 7, 11, ..., 83, 87

b 5, 8, 11, ..., 119, 122

c 90, 88, 86, ..., 16, 14

d 4, 9, 14, ..., 224, 229

e $x, 3x, 5x, \ldots, 35x$

f $a, a + d, a + 2d, \ldots, a + (n - 1)d$

Exercise D

1 Find **i** the 20th and **ii** the nth terms of the following arithmetic series:

a $2 + 6 + 10 + 14 + 18 \ldots$

b $4 + 6 + 8 + 10 + 12 + \ldots$

c $80 + 77 + 74 + 71 + \ldots$

d $1 + 3 + 5 + 7 + 9 + \ldots$

e $30 + 27 + 24 + 21 + \ldots$

f $2 + 5 + 8 + 11 + \ldots$

g $p + 3p + 5p + 7p + \ldots$

h $5x + x + (-3x) + (-7x) + \ldots$

2 Find the number of terms in the following arithmetic series:

a $5 + 9 + 13 + 17 + \ldots + 121$

b $1 + 1.25 + 1.5 + 1.75 \ldots + 8$

c $-4 + -1 + 2 + 5 \ldots + 89$

d $70 + 61 + 52 + 43 \ldots + -200$

e $100 + 95 + 90 + \ldots + (-1000)$

f $x + 3x + 5x \ldots + 153x$

3 The first term of an arithmetic series is 14. If the fourth term is 32, find the common difference.

4 Given that the 3rd term of an arithmetic series is 30 and the 10th term is 9 find a and d. Hence find which term is the first one to become negative.

5 In an arithmetic series the 20th term is 14 and the 40th term is −6. Find the 10th term.

6 The first three terms of an arithmetic series are $5x$, 20 and $3x$. Find the value of x and hence the values of the three terms.

7 For which values of x would the expression -8, x^2 and $17x$ form the first three terms of an arithmetic series?

> **Hint:** Question 6 – Find two expressions equal to the common difference and set them equal to each other.

(Exercise E is on the next page)

Exercise E

1 Find the sums of the following series:

a $3 + 7 + 11 + 15 + \ldots$ (20 terms)

b $2 + 6 + 10 + 14 + \ldots$ (15 terms)

c $30 + 27 + 24 + 21 + \ldots$ (40 terms)

d $5 + 1 + -3 + -7 + \ldots$ (14 terms)

e $5 + 7 + 9 + \ldots + 75$

f $4 + 7 + 10 + \ldots + 91$

g $34 + 29 + 24 + 19 + \ldots + -111$

h $(x + 1) + (2x + 1) + (3x + 1) + \ldots + (21x + 1)$

2 Find how many terms of the following series are needed to make the given sum:

a $5 + 8 + 11 + 14 + \ldots = 670$

b $3 + 8 + 13 + 18 + \ldots = 1575$

c $64 + 62 + 60 + \ldots = 0$

d $34 + 30 + 26 + 22 + \ldots = 112$

3 Find the sum of the first 50 even numbers.

4 Carol starts a new job on a salary of £20 000. She is given an annual wage rise of £500 at the end of every year until she reaches her maximum salary of £25 000. Find the total amount she earns (assuming no other rises), **a** in the first 10 years and **b** over 15 years.

5 Find the sum of the multiples of 3 less than 100. Hence or otherwise find the sum of the numbers less than 100 which are not multiples of 3.

6 James decides to save some money during the six-week holiday. He saves 1p on the first day, 2p on the second, 3p on the third and so on. How much will he have at the end of the holiday (42 days)? If he carried on, how long would it be before he has saved £100?

7 The first term of an arithmetic series is **4**. The sum to 20 terms is -15. Find, in any order, the common difference and the 20th term.

Exercise F

1 Rewrite the following sums using Σ notation:

a $4 + 7 + 10 + \ldots + 31$

b $2 + 5 + 8 + 11 + \ldots + 89$

c $40 + 36 + 32 + \ldots + 0$

d The multiples of 6 less than 100

2 Calculate the following:

a $\sum_{r=1}^{5} 3r$

b $\sum_{r=1}^{10} (4r - 1)$

c $\sum_{r=1}^{20} (5r - 2)$

d $\sum_{r=0}^{5} r(r + 1)$

3 For what value of n does $\sum_{r=1}^{n} (5r + 3)$ first exceed 1000?

4 For what value of n would $\sum_{r=1}^{n} (100 - 4r) = 0$?

Exercise G

1 Find the sum, for the given number of terms, of each of the following geometric series. Give decimal answers correct to 4 places.

(a) $2 + 6 + 18 + \dots$ 10 terms (b) $2 - 6 + 18 - \dots$ 10 terms

(c) $1 + \frac{1}{2} + \frac{1}{4} + \dots$ 8 terms (d) $1 - \frac{1}{2} + \frac{1}{4} - \dots$ 8 terms

(e) $3 + 6 + 12 + \dots$ 12 terms (f) $12 - 4 + \frac{4}{3} - \dots$ 10 terms

2 Find the sum of each of the following geometric series. Give numerical answers in parts (a) to (g) as rational numbers.

(a) $1 + 2 + 4 + \dots + 1024$ (b) $1 - 2 + 4 - \dots + 1024$

(c) $3 + 12 + 48 + \dots + 196\,608$ (d) $1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{512}$

(e) $1 - \frac{1}{3} + \frac{1}{9} - \dots - \frac{1}{19\,683}$ (f) $10 + 5 + 2.5 + \dots + 0.156\,25$

(g) $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots + \frac{1}{1024}$ (h) $1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n}$

(i) $16 + 4 + 1 + \dots + \frac{1}{2^{2n}}$ (j) $81 - 27 + 9 - \dots + \frac{1}{(-3)^n}$

3 Find the sum of each of the following geometric series. Give numerical answers as rational numbers.

(a) $\sum_{i=1}^{5} 3 \times 2^{i-1}$ (b) $\sum_{i=1}^{4} 2 \times (-2)^{i-1}$

(c) $\sum_{i=1}^{8} 16 \times \left(\frac{1}{2}\right)^{i-1}$ (d) $\sum_{i=1}^{7} 4^i$

4 A well-known story concerns the inventor of the game of chess. As a reward for inventing the game it is rumoured that he was asked to choose his own prize. He asked for 1 grain of rice to be placed on the first square of the board, 2 grains on the second square, 4 grains on the third square and so on in geometric progression until all 64 squares had been covered. Calculate the total number of grains of rice he would have received. Give your answer in standard form!

5 A problem similar to that of Question 4 is posed by the child who negotiates a pocket money deal of 1p on 1 February, 2p on 2 February, 4p on 3 February and so on for 28 days. How much should the child receive in total during February?

6 A firm sponsors a local orchestra for seven years. It agrees to give £2500 in the first year, and to increase its contribution by 20% each year. Show that the amounts contributed each year form a geometric sequence. How much does the firm give the orchestra altogether during the period of sponsorship?

7 An explorer sets out across the desert with 100 litres of water. He uses 6 litres on the first day. On subsequent days he rations himself to 95% of the amount he used the day before. Show that he has enough water to last for 34 days, but no more. How much will he then have left?

8 A competitor in a pie-eating contest eats one-third of his pie in the first minute. In each subsequent minute he eats three-quarters of the amount he ate in the previous minute. Find an expression for the amount of pie he has left to eat after n minutes. Show that he takes between 4 and 5 minutes to finish the pie.

9 Find expressions for the sum of n terms of the following series. Give your answer, in terms of n and x, in as simple a form as possible.

(a) $x + x^2 + x^3 + \cdots$ n terms
(b) $x - x^2 + x^3 - \cdots$ n terms
(c) $x + \frac{1}{x} + \frac{1}{x^3} + \cdots$ n terms
(d) $1 - \frac{1}{x^2} + \frac{1}{x^4} + \cdots$ n terms

Exercise H

1 Find the sum to infinity of the following geometric series. Give your answers to parts (a) to (j) as whole numbers, fractions or exact decimals.

(a) $1 + \frac{1}{2} + \frac{1}{4} + \ldots$
(b) $1 + \frac{1}{3} + \frac{1}{9} + \ldots$
(c) $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots$
(d) $0.1 + 0.01 + 0.001 + \ldots$
(e) $1 - \frac{1}{3} + \frac{1}{9} - \ldots$
(f) $0.2 - 0.04 + 0.008 - \ldots$
(g) $\frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \ldots$
(h) $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \ldots$
(i) $10 - 5 + 2.5 - \ldots$
(j) $50 + 10 + 2 + \ldots$
(k) $x + x^2 + x^3 + \ldots$, where $-1 < x < 1$
(l) $1 - x^2 + x^4 - \ldots$, where $x^2 < 1$
(m) $1 + x^{-1} + x^{-2} + \ldots$, where $x > 1$
(n) $x^2 - x + 1 - \ldots$, where $x > 1$

2 Express each of the following recurring decimals as exact fractions.

(a) 0.363 636...
(b) 0.123 123 123...
(c) 0.555...
(d) 0.471 471 471...
(e) 0.142 857 142 857 142 857...
(f) 0.285 714 285 714 285 714...
(g) 0.714 285 714 285 714 285...
(h) 0.857 142 857 142 857 142...

3 Find the common ratio of a geometric series which has a first term of 5 and a sum to infinity of 6.

4 Find the common ratio of a geometric series which has a first term of 11 and a sum to infinity of 6.

5 Find the first term of a geometric series which has a common ratio of $\frac{3}{4}$ and a sum to infinity of 12.

6 Find the first term of a geometric series which has a common ratio of $-\frac{3}{5}$ and a sum to infinity of 12.

7 Identify the general term u_i for the following geometric progressions. Also find $\sum_{i=1}^{10} u_i$ and, if the series is convergent, $\sum_{i=1}^{\infty} u_i$.

(a) $1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots$
(b) $2 + 4 + 8 + 16 + \ldots$
(c) $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots$
(d) $1 - \frac{1}{10} + \frac{1}{100} - \frac{1}{1000} + \ldots$

8.

A beetle starts at a point O on the floor and walks 0.6 m east, then 0.36 m west, 0.216 m east and so on. Find its final position and how far it actually walks.

9.

A 'supa-ball' is thrown upwards from ground level. It hits the ground after 2 seconds and continues to bounce. The time it is in the air for a particular bounce is always 0.8 of the time for the previous bounce. How long does it take for the ball to stop bouncing?